CITY PRODUCTS OF RIGHT-ANGLED BUILDINGS AND THEIR UNIVERSAL GROUPS

JENS BOSSAERT AND TOM DE MEDTS

In memory of Jacques Tits, architect of buildings

ABSTRACT. We introduce the notion of *city products* of right-angled buildings that produces a new right-angled building out of smaller ones. More precisely, if M is a right-angled Coxeter diagram of rank n and $\Delta_1, \ldots, \Delta_n$ are right-angled buildings, then we construct a new right-angled building $\Delta := \overline{X}_M(\Delta_1, \ldots, \Delta_n)$. We can recover the buildings $\Delta_1, \ldots, \Delta_n$ as residues of Δ , but we can also construct a *skeletal building* of type M from Δ that captures the large-scale geometry of Δ .

We then proceed to study universal groups for city products of right-angled buildings, and we show that the universal group of Δ can be expressed in terms of the universal groups for the buildings $\Delta_1, \ldots, \Delta_n$ and the structure of M. As an application, we show the existence of many examples of pairs of different buildings of the same type that admit (topologically) isomorphic universal groups, thereby vastly generalizing a recent example by Lara Beßmann.

1. INTRODUCTION

A building is called *right-angled* if its Coxeter group is right-angled, which means that the only values occurring in its Coxeter matrix are 1, 2 and ∞ . The proto-typical example is the case where the Coxeter matrix has rank 2 with a label ∞ , in which case the building is a tree. In general, the behavior of right-angled buildings is somewhat comparable to that of trees, but in a combinatorially much more complicated (and therefore much more interesting) way.

The first systematic study of right-angled buildings is by Frédéric Haglund and Frédéric Paulin [HP03], who showed the existence and uniqueness of right-angled buildings for any set of parameters (see Theorem 2.17 below). Later, right-angled buildings have been used to construct interesting examples of *lattices*, as in the work of Angela Kubena, Anne Thomas and Kevin Wortman [Tho06, TW11, KT12].

Our motivation for studying right-angled buildings, initiated by Pierre-Emmanuel Caprace in [Cap14], is the connection with totally disconnected locally compact groups. More precisely, the automorphism group of a right-angled building is always totally disconnected with respect to the permutation topology, and if the building is locally finite, then the automorphism group is also locally compact. This is not true in general, but these automorphism groups contain lots of interesting subgroups, namely so-called *universal groups*, that can still be locally compact even if the building is not locally finite.

These universal groups were first introduced and studied for trees by Marc Burger and Shahar Mozes in their seminal paper [BM00]. This concept has been generalized to right-angled buildings by the second author in joint work with Ana C. Silva and Koen Struyve in [DMdSS18, DMdS19] in the locally finite case, and has been further

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generalized and studied without this assumption in our paper [BDM21], focussing on topological properties.

For some right-angled buildings, the large-scale geometry looks like a tree; see, for instance, Fig. 1 below. This raises the question whether it is possible, in these cases, to somehow reverse the process, i.e., whether we can start from a tree and obtain a more complicated right-angled building by "inserting" more complicated blocks at each vertex of the tree.

This idea gave rise to the construction that we introduce and study in this paper. We call it the *city product* of buildings, as it is a way to construct larger objects out of a given number of buildings, guided by the rough structure of yet another right-angled diagram. More precisely, if M is a right-angled Coxeter diagram of rank n and $\Delta_1, \ldots, \Delta_n$ are right-angled buildings, then we construct a new rightangled building $\Delta := X_M(\Delta_1, \ldots, \Delta_n)$. We can recover the buildings $\Delta_1, \ldots, \Delta_n$ as residues of Δ , but we can also construct a *skeletal building* Φ of type M from Δ that captures the large-scale geometry of Δ . Constructing this building Φ is not difficult, but it turns out to be far from trivial to show that it is indeed a building. This is the content of Proposition 3.14, which relies on the new notions of *weak homotopies* and *parkour maps* that we have introduced for this purpose.

It turns out that the universal groups for these city products can be described as the universal group of this skeletal building Φ with respect to universal groups for each of the smaller buildings $\Delta_1, \ldots, \Delta_n$; this is the content of Theorem 4.1.

In a recent preprint [Beß22], Lara Beßmann has shown the existence of pairs of different right-angled buildings, both of type $\bullet^{\infty} \bullet^{\infty} \bullet$, admitting universal groups that are topologically isomorphic. Her method relies on the notion of tree-wall trees from [DMdSS18] and only works for star-shaped diagrams (see Examples 5.2(1)). We show that this can be interpreted in terms of city products, which allows us to produce many more examples of such pairs. This is the content of Theorem 5.1.

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2. Preliminaries

2.1. Coxeter systems.

Definition 2.1. (i) Let *I* be any index set and *M* a function

$$M: I \times I \to \mathbb{N} \cup \{\infty\}: (i, j) \mapsto m_{ij}$$

satisfying $m_{ii} = 1$, $m_{ij} \ge 2$, and $m_{ij} = m_{ji}$ for all $i \ne j \in I$. Then the Coxeter group of type M is the group defined by the presentation

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle.$$

When $m_{ij} = \infty$, this means that no relation on $s_i s_j$ should be imposed. Note that the assumption that $m_{ii} = 1$ for all $i \in I$ immediately implies that the generators s_i are involutions. Additionally, note that when $m_{ij} = 2$, the generators s_i and s_j commute.

Together with the generating set $S = \{s_i \mid i \in I\}$, the pair (W, S) is called the *Coxeter system of type M*. The rank of (W, S) is the cardinality of *I*.

We can represent M by means of its *Coxeter matrix* (m_{ij}) , or more commonly its *Coxeter diagram*: the nodes of the diagram are the elements of I

(sometimes with explicit labels), and two nodes are connected by a decorated edge according to the following rules:

$\overset{i}{\bullet}$	j_{ullet}	$\overset{i}{\bullet} \underbrace{\qquad \qquad }_{\bullet} \overset{j}{\bullet}$	$\overset{i}{\bullet} = \overset{j}{=} \bullet$	$\overset{i}{\bullet} \overset{m_{ij}}{-} \overset{j}{\bullet}$
m_{ij}	= 2	$m_{ij} = 3$	$m_{ij} = 4$	$m_{ij} \ge 5$

- (ii) We call a Coxeter system (W, S) *irreducible* if the underlying graph of its Coxeter diagram is connected, and *reducible* otherwise.
- (iii) We call a Coxeter system (W, S) right-angled if $m_{ij} \in \{2, \infty\}$ for all $i \neq j$.

In general, non-isomorphic Coxeter systems may have isomorphic Coxeter groups, but this cannot occur for right-angled Coxeter systems:

Theorem 2.2. If a right-angled Coxeter group W admits two Coxeter systems (W, S) and (W, S'), then these Coxeter systems are isomorphic (i.e. there is a diagram-preserving bijection $S \to S'$).

Proof. We refer to [Rad02] or [Hos03].

Definition 2.3. Let (W, S) be a Coxeter system over some index set I.

(i) We will write I^* for the free monoid over I. The elements of I^* will be called *words*.

(ii) There is a natural surjective evaluation morphism of monoids

$$\epsilon \colon I^* \to W \colon i \mapsto s_i.$$

Definition 2.4. For every $i \neq j$ such that m_{ij} is finite, define in I^* the word

$$p(i,j) = \begin{cases} (ij)^k & \text{if } m_{ij} = 2k \text{ is even,} \\ j(ij)^k & \text{if } m_{ij} = 2k+1 \text{ is odd.} \end{cases}$$

In other words, p(i, j) is the word with m_{ij} alternating letters *i* and *j*, ending in *j*. When $m_{ij} = \infty$, p(i, j) remains undefined.

Definition 2.5. Let $i, j \in I$ and $w_1, w_2 \in I^*$.

- (i) An elementary homotopy (or also a braid relation) is a transformation of a word $w_1 p(i, j) w_2$ into the word $w_1 p(j, i) w_2$.
- (ii) Two words w and w' are *homotopic* if w can be transformed into w' by a sequence of elementary homotopies; we denote this by $w \simeq w'$. Clearly, homotopy is an equivalence relation and preserves the length of the words.
- (iii) An elementary contraction is a transformation of a word $w_1 i i w_2$ into the word $w_1 w_2$.
- (iv) An elementary expansion is a transformation of a word $w_1 w_2$ into a word $w_1 ii w_2$.
- (v) A word is called *reduced* if it is not homotopic to a word of the form $w_1 ii w_2$ (for some $i \in I$).
- (vi) Two words w and w' are called *equivalent* if w can be transformed into w' by a sequence of elementary homotopies, contractions, and expansions. Clearly, every equivalence class contains some reduced word.
- **Theorem 2.6.** (i) Two words w and w' are equivalent if and only if $\epsilon(w) = \epsilon(w')$.
 - (ii) Two reduced words w and w' are equivalent if and only if they are homotopic.
 - (iii) Let w be a reduced word and let $i \in I$. If iw (or wi) is not reduced, then w is homotopic to a word that begins (or ends, respectively) with i.

Proof. By the defining relations $(s_i s_j)^{m_{ij}} = 1$ in the presentation, p(i, j) and p(j, i) have the same image under ϵ , and $\epsilon(ii)$ is the identity. Statement (i) follows immediately. For (ii), we refer to [Ron09, Theorem 2.11]. Statement (iii) is [Ron09, Corollary 2.13].

2.2. Chamber systems. Our approach is based on [Ron09].

Definition 2.7. Let I be any index set. A chamber system over I is a set Δ together with, for every $i \in I$, an equivalence relation called *i*-adjacency. The elements of Δ are called *chambers*. If two chambers c and d are *i*-adjacent, we write $c \sim_i d$, or simply $c \sim d$ if we do not want to stress the adjacency type. The cardinality |I| is called the rank of Δ . In this paper, the rank will always be finite.

We will usually say that " Δ is a chamber system" when the equivalence relations on Δ are clear from the context.

Definition 2.8. Let Δ be a chamber system over *I*. A gallery γ in Δ is a finite sequence of pairwise adjacent chambers

$$c_0 \sim_{i_1} c_1 \sim_{i_2} \cdots \sim_{i_n} c_n$$

for certain $i_1, \ldots, i_n \in I$. We call the word $i_1 \cdots i_n \in I^*$ the *type* of γ , and the integer *n* the *length* of γ . If there is no strictly shorter gallery from c_0 to c_n , then we call γ a *minimal* gallery.

Chamber systems come equipped with a natural metric

dist:
$$\Delta \times \Delta \to \mathbb{N} \cup \{\infty\}$$

defined by declaring dist(c, d) to be the minimal length of all galleries joining c and d (or ∞ if there is no such gallery). It is clear that this distance function is positive-definite, symmetric, and satisfies the triangle inequality.

Definition 2.9. Let $J \subseteq I$. A subset $C \subseteq \Delta$ is called *J*-connected if any two chambers in *C* can be joined by a gallery of type in J^* . A residue of type *J*, or simply a *J*-residue, is a *J*-connected component of Δ . A panel of type *j*, or simply a *j*-panel, is a residue of type $\{j\}$. The set of all *J*-residues of the chamber system Δ will be denoted by $\operatorname{Res}_J(\Delta)$.

Note that each J-residue is, in its own right, a connected chamber system over the index set J.

Definition 2.10. A chamber system is called *thin* if every panel contains exactly two chambers, and *thick* if every panel contains at least three chambers. (Panels containing only a single chamber are degenerate cases that should not occur in any reasonable application.)

Note that a chamber system might be neither thin nor thick.

Definition 2.11. A map $\varphi: \Delta_1 \to \Delta_2$ between two chamber systems is a morphism if $\varphi(c) \sim \varphi(d)$ in Δ_2 whenever $c \sim d$ in Δ_1 . As usual, an *isomorphism* is a bijective morphism, and an *automorphism* is an isomorphism to the same chamber system. Assuming that Δ_1 and Δ_2 have the same index set, a morphism is *type-preserving* if $\varphi(c) \sim_i \varphi(d)$ whenever $c \sim_i d$. In this paper, we shall always assume morphisms to be type-preserving.

The set of all automorphisms of a chamber system Δ forms a group, denoted by Aut(Δ).

Definition 2.12. Let (W, S) be a Coxeter system of type M over I. Define a chamber system over I with the elements of W as chambers, and declare two group elements v and w to be *i*-adjacent if and only if $vs_i = w$. The resulting chamber system is called the *Coxeter complex of type* M. Coxeter complexes are always

connected and thin: every chamber is *i*-adjacent to exactly one other chamber for every $i \in I$.

Observe that the Coxeter complex associated to a Coxeter system (W, S) is nothing more than the (undirected) Cayley graph of W with respect to the generating set S.

Theorem 2.13. A gallery in a Coxeter complex is minimal if and only if its type is reduced.

Proof. See [Ron09, Theorem 2.11].



FIGURE 1. A right-angled Coxeter complex

2.3. Right-angled buildings.

Definition 2.14. Let (W, S) be a Coxeter system of type M over some index set I. A building (Δ, δ) of type M is a chamber system Δ over I such that every panel contains at least two chambers, equipped with a map $\delta: \Delta \times \Delta \to W$ satisfying the following property for every reduced word $w \in I^*$:

 $\delta(c, d) = \epsilon(w)$ if and only if c and d can be joined by a gallery of type w.

Such a gallery is automatically minimal by Theorem 2.13. In particular, the distance between two chambers c and d is exactly the length of $\delta(c, d)$ in the word metric of W (w.r.t. generating set S).

The group W is called the Weyl group of the building, and the map δ is called the W-distance or Weyl distance function.

We shall usually identify the building with its chamber set and abbreviate (Δ, δ) to Δ .

Definition 2.15. A building Δ is called *right-angled* if its underlying Coxeter system (W, S) is right-angled (as defined in Definition 2.1(iii)).

Definition 2.16. A building Δ over I is called *semiregular with parameters* $(q_i)_{i \in I}$ if for each $i \in I$, all panels of type i have the same (possibly infinite) cardinality $q_i \geq 2$. Note that the thin buildings are precisely the semiregular buildings with parameters $q_i = 2$ for all i.

The following result is attributed to Haglund and Paulin, but they point out that this fact was already known to Mark Globus (but unpublished), Michael Davis and Gabor Moussong, and Tadeusz Januszkiewicz and Jacek Świątkowski.

Theorem 2.17. For any choice of (possibly infinite) cardinal numbers $(q_i)_{i \in I}$ with $q_i \geq 2$, there exists a semiregular right-angled building Δ with these parameters. Moreover, Δ is unique up to isomorphism, the automorphism group $\operatorname{Aut}(\Delta)$ acts transitively on the chambers, and every automorphism of a residue of Δ extends to an automorphism of Δ .

Proof. See [HP03, Proposition 1.2].

2.4. Colorings and implosions of right-angled buildings. In order to keep track of the local behavior of a building automorphism, it is useful to introduce colorings of the building. Throughout this section, Δ is a semiregular right-angled building with parameters $(q_i)_{i \in I}$. The following notion of legal colorings was introduced in [DMdSS18, Definition 2.42].

Definition 2.18. Consider a set Ω_i of cardinality q_i for each $i \in I$, the elements of which we call *i*-colors or *i*-labels. A legal coloring of Δ (with color sets Ω_i) is a map

$$\lambda \colon \Delta \to \prod_{i \in I} \Omega_i \colon c \mapsto (\lambda_i(c))_{i \in I}$$

satisfying the following properties for every $i \in I$ and for every *i*-panel \mathcal{P} :

- (i) the restriction $\lambda_i|_{\mathcal{P}} \colon \mathcal{P} \to \Omega_i$ is a bijection;
- (ii) for every $j \neq i$, the restriction $\lambda_j|_{\mathcal{P}} \colon \mathcal{P} \to \Omega_j$ is a constant map.

Such a legal coloring is essentially unique:

Proposition 2.19. Let λ and λ' be two legal colorings of a right-angled building Δ using identical color sets. Let c and c' be two chambers such that $\lambda(c) = \lambda'(c')$. Then there exists an automorphism $g \in \operatorname{Aut}(\Delta)$ such that g.c = c' and $\lambda' \circ g = \lambda$.

Proof. See [DMdSS18, Proposition 2.44].

We now recall the notion of an implosion of a right-angled building, introduced in [BDM21, Definition 5.2].

Definition 2.20. Let Δ be a semiregular right-angled building over I and let λ be a legal coloring of Δ (using color sets Ω_i). For each $i \in I$, consider an equivalence relation \equiv_i on Ω_i , let $\Omega'_i := \Omega_i / \equiv_i$ and set $q'_i := |\Omega'_i|$. For each $\lambda_i \in \Omega_i$, we write $[\lambda_i]$ for the corresponding element of Ω'_i . Let

 $I' = \{i \in I \mid \equiv_i \text{ is not the universal relation}\} = \{i \in I \mid q'_i \neq 1\}.$

Define a new semiregular right-angled building Δ' over I' with diagram induced by the diagram of Δ , with parameters q'_i (for every $i \in I'$), and with a legal coloring λ' using the quotient Ω'_i as the set of *i*-colors.

Recall that a map $f: X \to Y$ between metric spaces is called *nonexpansive* if it does not increase distances, i.e., if $\operatorname{dist}_Y(f(x_1), f(x_2)) \leq \operatorname{dist}_X(x_1, x_2)$ for every pair (x_1, x_2) of points in X.

Proposition 2.21. Let Δ , λ , \equiv_i and Δ' be as in Definition 2.20. Let $c_0 \in \Delta$ be any chamber and let $c'_0 \in \Delta'$ be such that $\lambda'_i(c'_0) = [\lambda_i(c_0)]$ for every $i \in I'$. Then there exists a unique nonexpansive epimorphism τ of chamber systems from Δ onto Δ' mapping c_0 to c'_0 such that $\lambda'_i(\tau(c)) = [\lambda_i(c)]$ for all $c \in \Delta$.

Proof. See [BDM21, Proposition 5.1 and Remark 5.4].

Definition 2.22. We call the pair (Δ', τ) from Proposition 2.21 the *implosion* of Δ with *centre* c_0 (with respect to the relations \equiv_i).

Corollary 2.23. Let Δ be a semiregular right-angled building of type M over I, let $J \subseteq I$, and let Γ be the semiregular building of type M_J over J with the same parameters as Δ . Then there is a map $\varphi_J \colon \Delta \to \Gamma$ with the following properties:

- (i) for every residue \mathcal{R} of type J, the restriction $\varphi_J|_{\mathcal{R}}$ is an isomorphism;
- (ii) for every residue \mathcal{R} of type $I \setminus J$, the restriction $\varphi_J|_{\mathcal{R}}$ is a constant map.

Proof. This follows from Proposition 2.21 by taking as equivalence relations \equiv_i either the equality relation if $i \in J$ or the universal relation if $i \notin J$.

2.5. Universal groups. Universal groups for right-angled buildings were first introduced in [DMdSS18] and further studied in the locally finite case in [DMdS19]. Their topological properties in the general case have been further investigated in [BDM21].

Definition 2.24. Let Δ be a semiregular right-angled building over I, with parameters $(q_i)_{i \in I}$. For each i, let Ω_i be a color set of size q_i and let λ be a corresponding legal coloring of Δ .

(i) Consider an automorphism $g \in Aut(\Delta)$ and an arbitrary *i*-panel \mathcal{P} . Then we define the *local action of* g at \mathcal{P} as the map

$$\sigma_{\lambda}(g,\mathcal{P}) = \lambda_i \big|_{g\mathcal{P}} \circ g \big|_{\mathcal{P}} \circ \lambda_i \big|_{\mathcal{P}}^{-1},$$

which is a permutation of Ω_i by definition of λ . In other words, the local action $\sigma_{\lambda}(g, \mathcal{P})$ is the map that makes the following diagram commute.

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{g} & g\mathcal{P} \\ & \downarrow^{\lambda_i} & \downarrow^{\lambda_i} \\ \Omega_i & \xrightarrow{\sigma_{\lambda}(g,\mathcal{P})} & \Omega_i \end{array}$$

- (ii) Let \mathbf{F} be a collection of permutation groups $F_i \leq \text{Sym}(\Omega_i)$, indexed by $i \in I$. The universal group of \mathbf{F} over Δ (with respect to λ) is the group
- $\mathcal{U}^{\lambda}_{\Delta}(\mathbf{F}) = \{g \in \operatorname{Aut}(\Delta) \mid \sigma_{\lambda}(g, \mathcal{P}) \in F_i \text{ for each } i \in I \text{ and each } \mathcal{P} \in \operatorname{Res}_i(\Delta) \}.$

In words, $\mathcal{U}^{\lambda}_{\Delta}(\mathbf{F})$ is the group of automorphisms that locally act like permutations in F_i . We hence call the groups F_i the *local groups* and we refer to the collection \mathbf{F} as the *local data* for the universal group.

- **Remark 2.25.** (i) When the coloring λ is clear from the context, we will usually omit the explicit reference to λ and simply use the notation $\sigma(g, \mathcal{P})$ and $\mathcal{U}_{\Delta}(\mathbf{F})$ instead. In fact, the choice of λ is irrelevant, since different colorings give rise to conjugate subgroups of Aut(Δ); see [DMdSS18, Proposition 3.7(1)]*.
 - (ii) When each of the groups F_i in the local data F is given as a permutation group acting on some set Ω_i which is clear from the context, then we will also use the notation $\mathcal{U}_M(F)$ for $\mathcal{U}_\Delta(F)$, where Δ is then the unique right-angled building of type M over I with parameters $(|\Omega_i|)_{i \in I}$.

^{*}The statement of [DMdSS18, Proposition 3.7(1)] is for *locally finite* right-angled buildings only, but the proof continues to hold for arbitrary right-angled buildings, as pointed out already in [BDM21, §2.3].

The universal groups come equipped with a natural topology, namely the *permutation topology*, which is defined by taking as an identity neighborhood basis the collection of all pointwise stabilizers of finite subsets of Δ .

The following observation is worth mentioning, because this is precisely the type of result we will be generalizing later.

Lemma 2.26. Let Δ be a reducible right-angled building Δ over I. Let J_1, \ldots, J_m be the connected components of the diagram of Δ . Then the universal group $\mathcal{U}_{\Delta}(\mathbf{F})$ splits as a direct product

$$\mathcal{U}_{\Delta}(\mathbf{F}) \cong \mathcal{U}_{\mathcal{R}_1}(\mathbf{F}|_{J_1}) \times \cdots \times \mathcal{U}_{\mathcal{R}_m}(\mathbf{F}|_{J_m}),$$

where each \mathcal{R}_{ℓ} is a residue of type J_{ℓ} .

Proof. Since Δ is isomorphic to the direct product $\mathcal{R}_1 \times \cdots \times \mathcal{R}_m$ and has automorphism group $\operatorname{Aut}(\Delta) \cong \operatorname{Aut}(\mathcal{R}_1) \times \cdots \times \operatorname{Aut}(\mathcal{R}_m)$, this follows immediately from the definition.

3. City products

In this section, we develop a construction for creating new right-angled buildings of a higher rank by gluing together lower rank buildings along another diagram. Our construction is inspired by the observation that the large-scale geometry of certain right-angled buildings (such as Fig. 1) resembles that of a tree; the city product structure explains this behavior in a broad sense.

3.1. Weak homotopies. We start with some combinatorics, the goal of which will become clear later on.

Definition 3.1. Let $i, j \in I$ with $m_{ij} = 2$ and define the set

 $P(i,j) = \{ w \in \{i,j\}^* \mid w \text{ contains at least one } i \text{ and at least one } j \}.$

A weak homotopy is a transformation of a word $w_1 p w_2$ into a word $w_1 p' w_2$ where $w_1, w_2 \in I^*$ and $p, p' \in P(i, j)$. Two words w and w' are weakly homotopic if w can be transformed into w' by a sequence of weak homotopies.

Definition 3.2. Let \prec be a total order on I. Endow I^* with the induced lexicographical order. Then every word $w \in I^*$ is homotopic to a unique lexicographically minimal word that we call the *normal form* of w.

Proposition 3.3. Let (W, S) be a right-angled Coxeter system over I and let \prec be a total order on I.

- (i) If two words are homotopic, then their normal forms are equal.
- (ii) A word is reduced if and only if its normal form contains no consecutive duplicate letters.
- (iii) The normal forms of weakly homotopic words are equal up to consecutive duplicate letters.

Proof. Claim (i) follows immediately from the definitions. For (ii), let $w \simeq w_1 i i w_2$ and assume that the normal form contains no subword *ii*. Mark the two letters *i* in $w_1 i i w_2$ and write the normal form as $n_1 i n_2 i n_3$ (where the two letters *i* are the marked ones). Then n_2 is not the empty word, so let *k* be its first letter; by assumption, $k \neq i$. By homotopy, all letters in n_2 are contained in $\{i\} \cup$ $\{i\}^{\perp}$, where $\{i\}^{\perp} := \{j \in I \mid m_{ij} = 2\}$. It follows that the normal form cannot be lexicographically minimal: if $i \prec k$, then the homotopic word $n_1 i i n_2 n_3$ is lexicographically smaller, and if $i \succ k$, then $n_1 n_2 i i n_3$ is smaller. Claim (ii) follows. For claim (iii), it suffices to note that the effect of a weak homotopy of a word on its normal form is that a subword $i^m j^n$ with $m \ge 1$, $n \ge 1$, is replaced by another such word.

Corollary 3.4. Let (W, S) be a right-angled Coxeter system over I and let \prec be a total order on I. If two reduced words $w, w' \in I^*$ are weakly homotopic, then they are homotopic.

Proof. Letting \prec be any total order, this follows readily from Proposition 3.3(ii) and (iii).

3.2. City product of diagrams. Now let us go back to the building realm and define an operation on the diagrams first.

Definition 3.5. Let M be a diagram of rank n over the index set $\{1, \ldots, n\}$, and for each $\ell \in \{1, \ldots, n\}$, let M_{ℓ} be a diagram over an index set I_{ℓ} . Then we define a new diagram as follows:

- (i) the index set is the disjoint union $I = \bigsqcup_{\ell=1}^{n} I_{\ell}$;
- (ii) for every pair of elements $i \in I_{\ell}$ and $i' \in I_{\ell'}$, we set

$$m_{ii'} := \begin{cases} m_{ii'} \text{ (considered in } M_{\ell}) & \text{ if } \ell = \ell'; \\ m_{\ell\ell'} \text{ (considered in } M) & \text{ if } \ell \neq \ell'. \end{cases}$$

We call this the *city product* of the diagrams M_1, \ldots, M_n over M and denote it by $\Xi_M(M_1, \ldots, M_n)$. Clearly its rank is $\sum_{\ell=1}^n |I_\ell|$.

Notice that the special case of a city product over an edgeless diagram (i.e., $m_{ij} = 2$ for all $1 \le i \ne j \le n$) results in nothing more than the disjoint union of the diagrams M_1, \ldots, M_n . Two more examples are given in Fig. 2. (More examples will occur in Examples 5.2 later.) Our choice for the symbol Σ for city products is inspired by the example from Fig. 2(b).



FIGURE 2. City products of diagrams

Lemma 3.6. The diagram $X_M(M_1, \ldots, M_n)$ with $n \ge 2$ is irreducible if and only if M is irreducible.

Proof. This follows immediately from the definition.

3.3. City product of right-angled buildings. We can now continue to define city products of right-angled buildings.

Definition 3.7. Let M be a right-angled diagram over the index set $\{1, \ldots, n\}$, and for each $\ell \in \{1, \ldots, n\}$, let Δ_{ℓ} be a semiregular right-angled building of type M_{ℓ} over I_{ℓ} . Then we define the *city product* of the buildings $\{\Delta_1, \ldots, \Delta_n\}$ over M as follows:

- (i) the index set is the disjoint union $I = \bigsqcup_{\ell=1}^{n} I_{\ell}$;
- (ii) the (right-angled) diagram is the city product of diagrams $X_M(M_1, \ldots, M_n)$;
- (iii) for each $i \in I$, the parameter q_i of the new building is the parameter q_i of Δ_{ℓ} , where $i \in I_{\ell}$.

By Theorem 2.17, this defines a unique semiregular right-angled building (up to isomorphism), that we denote by $\Xi_M(\Delta_1, \ldots, \Delta_n)$. It will be convenient to define $\ell(i)$ (for $i \in I$) as the unique number in $\{1, \ldots, n\}$ such that $i \in I_{\ell(i)}$.

Note that for each $\ell \in \{1, \ldots, n\}$, the residues of type $I_{\ell} \subseteq I$ of the city product $\Xi_M(\Delta_1, \ldots, \Delta_n)$ are isomorphic to the original building Δ_{ℓ} . As a special case of Corollary 2.23, we then obtain:

Lemma 3.8. Let $\Delta = \Xi_M(\Delta_1, \dots, \Delta_n)$ be a city product. Then for each $\ell \in \{1, \dots, n\}$, there is a map $\varphi_\ell \colon \Delta \to \Delta_\ell$ with the following properties:

- (i) for each residue \mathcal{R} of type I_{ℓ} , the restriction $\varphi_{\ell}|_{\mathcal{R}} \colon \mathcal{R} \to \Delta_{\ell}$ is an isomorphism;
- (ii) for each residue \mathcal{R} of type $I \setminus I_{\ell}$, the restriction $\varphi_{\ell}|_{\mathcal{R}} : \mathcal{R} \to \Delta_{\ell}$ is a constant map.

Proof. This follows immediately from Corollary 2.23.

We can then easily lift colorings of the subbuildings to a coloring of the full city product.

Lemma 3.9. Let $\Delta = \Xi_M(\Delta_1, \ldots, \Delta_n)$ be a city product. For each $\ell \in \{1, \ldots, n\}$, let λ^{ℓ} be a legal coloring of Δ_{ℓ} with color sets Ω_i (where i ranges over I_{ℓ}). Then the collection of maps

$$\lambda_i' = \lambda_i^{\ell(i)} \circ \varphi_{\ell(i)}$$

provides a legal coloring of Δ with color sets Ω_i (where i ranges over $I = \bigsqcup_{\ell=1}^n I_\ell$).

Proof. This follows immediately from Lemma 3.8 and the definition of legal colorings. $\hfill \Box$

The city product construction over a diagram M essentially glues together smaller rank buildings as if they were chambers of a building of type M, hence the fact that the original buildings reemerge locally as residues (Lemma 3.8) should not be surprising. However, we can also recover a building of type M at the global scale by relaxing the adjacencies.

Definition 3.10. Let $\Delta = \Xi_M(\Delta_1, \ldots, \Delta_n)$ be a city product, where M is a rightangled diagram over $\{1, \ldots, n\}$. The *skeletal building* of Δ is the chamber system Φ over the index set $\{1, \ldots, n\}$ with the same chamber set as Δ , but with coarser adjacencies: we declare two chambers $c, d \in \Delta$ to be ℓ -adjacent in Φ if and only if they lie in the same residue of type I_ℓ in Δ .

We will prove in Proposition 3.14 that the skeletal building of a city product is, in fact, a building. First, we need an auxiliary definition and some combinatorial lemmas, laying the bridge between city products and weak homotopies.

Definition 3.11. Let $I = \bigsqcup_{\ell=1}^{n} I_{\ell}$.

(i) The parkour map of (I_1, \ldots, I_n) is the map

 $r\colon I^*\to \{1,\ldots,n\}^*$

that first replaces every letter $i \in I$ by $\ell(i) \in \{1, \ldots, n\}$ and then removes consecutive duplicates (i.e., replaces them by a single letter).

- (ii) The map $\{1, \ldots, n\}^* \to \{1, \ldots, n\}^*$ that replaces consecutive duplicates by a single letter will be denoted by $v \mapsto \overline{v}$.
- (iii) The maximal subwords of a word $w \in I^*$ with letters in a common subset I_{ℓ} are called the *blocks* of w. These are precisely the maximal subwords such that the image under r is a single letter.

Example 3.12. Consider the index sets

$$I_1 = \{1_a, 1_b, 1_c\}, \quad I_2 = \{2_a, 2_b\}, \quad I_3 = \{3_a, 3_b, 3_c\}, \quad I = I_1 \cup I_2 \cup I_3.$$

Then for the word $w = 2_a 2_b 3_c 1_c 1_a 1_b 1_a 3_b$, the image is r(w) = 2313. The blocks are the words

$$2_a 2_b, \quad 3_c, \quad 1_c 1_a 1_b 1_a, \quad 3_b$$

The interpretation in terms of the skeletal building is now clear: Let $\Delta = \Sigma_M(\Delta_1, \ldots, \Delta_n)$ be a city product of type $I = \bigsqcup_{\ell=1}^n I_\ell$ and let Φ be its skeletal building. If w is the type of a gallery in Δ , then r(w) is the type of a gallery in Φ with the same extremities, but replacing subgalleries in residues of type I_ℓ by a single jump of type ℓ . (This behavior explains our choice for the terminology "parkour map".)

When viewed as elements of the corresponding Coxeter groups, the interplay between words in I^* and words in $\{1, \ldots, n\}^*$ is not completely trivial — especially when considering reduced words. As illustrated in Fig. 3, images of reduced words under the parkour map are not necessarily reduced, nor are images of equivalent words necessarily equivalent.



(a) The ambient (reducible) city product. Notice that this occurs as a residue of the irreducible city product from Fig. 2(b).



(b) The parkour map $r: \{1_a, 1_b, 2_a, 2_b\}^* \to \{1, 2\}^*$. Horizontal lines alternate between 1_a and 1_b , vertical lines alternate between 2_a and 2_b .

FIGURE 3. The effect of the parkour map on equivalent types of minimal galleries

The following slightly technical lemma explains the connection in more detail.

Lemma 3.13. Let M be a diagram of rank n over the index set $\{1, \ldots, n\}$ and for each $\ell \in \{1, \ldots, n\}$, let M_{ℓ} be a diagram over I_{ℓ} . Consider the city product $\Xi_M(M_1, \ldots, M_n)$, with index set $I = \bigsqcup_{\ell=1}^n I_{\ell}$. Let $u \in I^*$ and let $r: I^* \to \{1, \ldots, n\}^*$ be the parkour map.

(i) If u ≃ u', then r(u) and r(u') are weakly homotopic (in the sense of Definition 3.1).

(ii) Assume that we have a homotopy $r(u) \simeq v$. Then there exists $u' \in I^*$ such that $u' \simeq u$ and $r(u') = \overline{v}$, where \overline{v} is as in Definition 3.11(ii).



- (iii) If u is reduced, then all blocks of u are reduced.
- (iv) If all blocks of u are reduced and r(u) is reduced, then u is reduced.
- (v) If $u \simeq u'$ and both r(u) and r(u') are reduced, then $r(u) \simeq r(u')$.

Proof. (i) Consider an elementary homotopy $u = u_1 i j u_2 \simeq u_1 j i u_2$. If $\ell(i) = \ell(j)$, then the image under r remains unchanged. Assume now that $\ell(i) \neq \ell(j)$

- $\ell(j)$. We distinguish three cases for the subword u_1 of u:
- **[L.a]:** u_1 is nonempty and the last letter of u_1 is in $I_{\ell(i)}$,
- **[L.b]:** u_1 is nonempty and the last letter of u_1 is in $I_{\ell(j)}$,
- **[L.c]:** u_1 is the empty word, or the last letter of u_1 is neither in $I_{\ell(i)}$ nor $I_{\ell(j)}$. Analogously, we distinguish three cases for u_2 :
- **[R.a]:** u_2 is nonempty and the first letter of u_2 is in $I_{\ell(i)}$,
- **[R.b]:** u_2 is nonempty and the first letter of u_2 is in $I_{\ell(j)}$,
- **[R.c]:** u_2 is the empty word, or the first letter of u_2 is neither in $I_{\ell(i)}$ nor $I_{\ell(j)}$. Depending on the nine combinations of possibilities, the elementary homotopy $u_1 ij u_2 \simeq u_1 ji u_2$ transforms the image r(u) by substituting some subword in $\{ij, ji, iji, jij, jijj, jiji\}$ (where we have simply written *i* and *j* instead of $\ell(i)$ and $\ell(j)$ for better readability) into another such word; see Table 1. In each case, the result is weakly homotopic to r(u).



	R.a	R.b	R.c
L.a	$iji \rightsquigarrow iji$	$ij \rightsquigarrow ijij$	$ij \rightsquigarrow iji$
L.b	$jiji \rightsquigarrow ji$	$jij \rightsquigarrow jij$	$jij \rightsquigarrow ji$
L.c	$iji \rightsquigarrow ji$	$ij \rightsquigarrow jij$	$ij \rightsquigarrow ji$

TABLE 1. The effect of an elementary homotopy on the image of the parkour map. We have simply written i and j instead of $\ell(i)$ and $\ell(j)$ for better readability.

(ii) Consider an elementary homotopy $r(u) = v_1 \ell_1 \ell_2 v_2 \simeq v_1 \ell_2 \ell_1 v_2 = v$ with $1 \leq \ell_1 \neq \ell_2 \leq n$ and such that ℓ_1 and ℓ_2 commute in M. Then we can write $u = u_1 b_1 b_2 u_2$, where b_1 and b_2 are the blocks corresponding to ℓ_1 and ℓ_2 , respectively. Since b_1 and b_2 have only letters in I_{ℓ_1} and in I_{ℓ_2} , which are sets of pairwise commuting generators in the Coxeter system, we have a homotopy $u' = u_1 b_2 b_1 u_2 \simeq u_1 b_1 b_2 u_2$ satisfying $r(u') = \overline{v}$. The claim now follows by induction on the number of elementary homotopies needed to go from r(u) to v.

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- (iii) This is obvious since any subword of a reduced word is reduced.
- (iv) Assume by means of contraposition that every block of u is reduced while u is not, i.e., there is a homotopy $u \simeq w_1 i i w_2$. Mark these two letters i and let b_1 and b_2 be the two blocks of u containing these two marked letters. Since every block is reduced, we have $b_1 \neq b_2$, hence $u = u_1 b_1 u_2 b_2 u_3$ such that u_2 is nonempty and $m_{ij} = 2$ for every letter j in u_2 . The image then satisfies $r(u) = r(u_1) \ell(i) r(u_2) \ell(i) r(u_3)$. By construction, $\ell(i)$ commutes with every letter in $r(u_2)$. Hence r(u) is not reduced.
- (v) This follows from (i) and Corollary 3.4.

Proposition 3.14. Let M be a diagram of rank n, let $\Delta_1, \ldots, \Delta_n$ be right-angled buildings, let $\Delta := \Xi_M(\Delta_1, \ldots, \Delta_n)$ be their city product and let Φ be its skeletal building. Then

- (i) Φ is a right-angled building of type M over $\{1, \ldots, n\}$;
- (ii) Φ is semiregular with parameters $q_{\ell} = |\Delta_{\ell}|$ for every $\ell \in \{1, \ldots, n\}$;
- (iii) ℓ -panels of Φ (as sets of chambers) are I_{ℓ} -residues of Δ and vice versa;
- (iv) the maps φ_{ℓ} with $\ell \in \{1, \ldots, n\}$ (introduced in Lemma 3.8) provide a legal coloring of Φ with color sets Δ_{ℓ} .

Proof. The only nontrivial claim is that Φ is indeed a building of type M; the other claims will then follow immediately from the definitions. Let us first write out the Weyl distance function in Φ and then verify that it satisfies the necessary properties.

Denote by W_{Δ} the Weyl group of the building Δ . Recall from Definition 2.3 the evaluation morphism $\epsilon_{\Delta} \colon I^* \to W_{\Delta}$. On the other hand, we have a Coxeter group W_{Φ} of type M, together with an evaluation morphism $\epsilon_{\Phi} \colon \{1, \ldots, n\}^* \to W_{\Phi}$. Next, we let $s \colon W_{\Delta} \to I^*$ be a section of ϵ_{Δ} with reduced images such that for each $w \in W_{\Delta}$, the word length |r(s(w))| is minimal among all possible choices for s(w). We claim that for each $w \in W_{\Delta}$, the word r(s(w)) in $\{1, \ldots, n\}^*$ is reduced. Indeed, if r(s(w)) would not be reduced, then by Lemma 3.13(ii), we would find some word $u' \in I^*$ with $u' \simeq s(w)$ (so that $\epsilon_{\Delta}(u') = \epsilon_{\Delta}(s(w))$) for which r(u') has smaller word length than r(s(w)).

Finally, we can define

$$\delta_{\Phi} = \epsilon_{\Phi} \circ r \circ s \circ \delta_{\Delta} \colon \Phi \times \Phi \to W_{\Phi}.$$

(Notice that the composition $\epsilon_{\Phi} \circ r \circ s$ is a map $W_{\Delta} \to W_{\Phi}$ but by no means a group homomorphism.)



Clearly, panels of Φ contain at least two chambers, since every such panel of Φ contains a panel of Δ . Now consider a reduced word v in $\{1, \ldots, n\}^*$. Our goal is to show that $\delta_{\Phi}(c, d) = \epsilon_{\Phi}(v)$ if and only if there exists a gallery of type v from c to d in Φ .

First, assume that $\delta_{\Phi}(c, d) = \epsilon_{\Phi}(v)$. By definition of δ_{Φ} , this means that the words $(r \circ s \circ \delta_{\Delta})(c, d)$ and v are equivalent. Moreover, both words are reduced, and hence homotopic by Theorem 2.6(ii). By Lemma 3.13(ii), this homotopy can be realized in I^* , i.e., we can find a word $u \in I^*$ such that $u \simeq (s \circ \delta_{\Delta})(c, d)$ and r(u) = v. The homotopy $u \simeq (s \circ \delta_{\Delta})(c, d)$ yields that u is reduced, hence by the

building axioms for Δ , there is a minimal gallery in Δ of type u from c to d. Then r(u) = v is the type of a gallery in Φ from c to d.

Conversely, assume that γ is a gallery of type v from c to d in Φ . We can "lift" γ to a gallery $\overline{\gamma}$ in Δ with the same extremities, by replacing each ℓ -adjacency in γ by a minimal gallery in a residue of type I_{ℓ} of Δ . Let \overline{v} be the type of $\overline{\gamma}$. Note that $r(\overline{v}) = v$ and that \overline{v} is reduced by Lemma 3.13(iv). Hence, we have $\delta_{\Delta}(c, d) = \epsilon_{\Delta}(\overline{v})$, so that $s(\delta_{\Delta}(c, d))$ and \overline{v} are homotopic by Theorem 2.6(i) and (ii). Then by Lemma 3.13(v), the images $(r \circ s \circ \delta_{\Delta})(c, d)$ and $r(\overline{v}) = v$ are homotopic, so that finally

$$\delta_{\Phi}(c,d) = (\epsilon_{\Phi} \circ r \circ s \circ \delta_{\Delta})(c,d) = \epsilon_{\Phi}(v).$$

This concludes our proof that Φ is a right-angled building of type M.

Example 3.15. Consider the Coxeter complex from Fig. 1. This is a thin building of rank three, so each of the three parameters q_1 , q_2 and q_3 is equal to 2. We can now view this building as a city product of two thin buildings, one of rank 1 and one of rank 2:



The skeletal building Φ of this city product is now obtained by merging the adjacencies "red" and "blue" together to a new ℓ_1 -adjacency; the adjacency "green" is unchanged and is our new ℓ_2 -adjacency. By doing so, the residues of type {red, blue} in Fig. 1 have now become panels (of type ℓ_1), and as a result, Φ is now a building of rank two, which is no longer a thin building: it is a semiregular tree with parameters $q_1 = 4$ and $q_2 = 2$; see Fig. 4. (Notice that this is a tree viewed as a *chamber complex*: the vertices of the corresponding tree correspond to the *panels* of this chamber complex.)



FIGURE 4. The skeletal building of the right-angled Coxeter complex from Fig. 1 viewed as city product. The grey blocks are panels of size 4, the green lines are panels of size 2.

Remark 3.16. The city product of right-angled diagrams can be interpreted in a purely graph-theoretical way, as follows. A subset X of the vertex set of a graph is called a *module* if it has the property that every vertex $v \notin X$ is either adjacent to

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all vertices in X or adjacent to no vertex in X. A graph is called *prime* if it has no non-trivial modules.

A right-angled diagram is then a non-trivial city product of lower rank rightangled diagrams if and only if its underlying graph is not prime. A decomposition of a right-angled diagram as a non-trivial city product of lower rank right-angled diagrams then corresponds to a *modular partition* of the underlying graph.

4. Universal groups of city products

Earlier in Lemma 2.26, we already observed that the universal group construction behaves nicely with respect to disjoint unions of diagrams. This operation on diagrams is a special case of the city product from Section 3 (over a diagram with only isolated nodes). In this section, we generalize Lemma 2.26 to arbitrary city products. More precisely, we will show that the universal group over a city product of buildings $\underline{X}_M(\Delta_1, \ldots, \Delta_n)$ is isomorphic to the universal group over the skeletal building of the universal groups over the buildings Δ_i .

Theorem 4.1. Let M be a right-angled diagram of rank n. For each $l \in \{1, ..., n\}$, let Δ_{ℓ} be a semiregular right-angled building of type M_{ℓ} over I_{ℓ} , equipped with a legal coloring λ^{ℓ} with color sets Ω_i (indexed by $i \in I_{\ell}$). Let $\Delta := \Xi_M(\Delta_1, ..., \Delta_n)$ be their city product over $I = \bigsqcup_{\ell=1}^n I_{\ell}$ and let Φ be its skeletal building over $\{1, ..., n\}$.

Assume that for each $i \in I$, we have a permutation group $F_i \leq \text{Sym}(\Omega_i)$, giving rise to local data \mathbf{F} over I. These local data restrict to local data \mathbf{F}_{ℓ} over I_{ℓ} for each $\ell \in \{1, \ldots, n\}$.

Then we have an isomorphism of topological groups

$$\mathcal{U}_{\Delta}(\mathbf{F}) \cong \mathcal{U}_{\Phi}\Big(\Big(\mathcal{U}_{\Delta_{\ell}}(\mathbf{F}_{\ell})\Big)_{\ell \in \{1,...,n\}}\Big).$$

Proof. Equip Δ with the coloring λ' from Lemma 3.9, assigning colors in the sets Ω_i (indexed by $i \in I$). Also equip its skeletal building Φ with the coloring φ from Proposition 3.14, assigning colors in the sets Δ_{ℓ} (indexed by $\ell \in \{1, \ldots, n\}$). Let \mathbf{F}' be the local data over $\{1, \ldots, n\}$ defined by

$$F'_{\ell} := \mathcal{U}_{\Delta_{\ell}}(\mathbf{F}_{\ell}) \quad \text{for each } \ell \in \{1, \dots, n\}.$$

First, every automorphism of Δ induces an automorphism of its skeletal building, hence we have a natural monomorphism

$$\iota \colon \operatorname{Aut}(\Delta) \hookrightarrow \operatorname{Aut}(\Phi).$$

Let $g \in \mathcal{U}_{\Delta}(\mathbf{F}) \leq \operatorname{Aut}(\Delta)$, let \mathcal{R} be any panel of Φ of type ℓ , and consider the local action of $\iota(g)$ as an automorphism of Φ at the panel \mathcal{R} . For readability, we will identify g with its image $\iota(g)$. We can also identify \mathcal{R} with a residue of Δ of type I_{ℓ} (which is isomorphic to Δ_{ℓ}). Then the local action

$$\sigma_{\varphi}(g,\mathcal{R}) = \varphi_{\ell} \big|_{g,\mathcal{R}} \circ g \big|_{\mathcal{R}} \circ \varphi_{\ell} \big|_{\mathcal{R}}^{-1}$$

is the composition of three isomorphisms $\Delta_{\ell} \to \mathcal{R} \to g.\mathcal{R} \to \Delta_{\ell}$ and is hence an automorphism of the building Δ_{ℓ} . It thus makes sense to consider the local action of $\sigma_{\varphi}(g,\mathcal{R})$ at an *i*-panel \mathcal{P} of Δ_{ℓ} with $i \in I_{\ell}$. Let $\mathcal{P}' = \varphi_{\ell}|_{\mathcal{R}}^{-1}(\mathcal{P})$, which is an *i*-panel of Δ by Lemma 3.8. We then have $\mathcal{P}' \subseteq \mathcal{R} \subseteq \Delta$, and the map $\varphi_{\ell} \colon \Delta \to \Delta_{\ell}$ restricts to isomorphisms $\varphi_{\ell}|_{\mathcal{R}} \colon \mathcal{R} \to \Delta_{\ell} \text{ and } \varphi_{\ell}|_{\mathcal{P}'} \colon \mathcal{P}' \to \mathcal{P}.$ Then

$$\begin{split} \sigma_{\lambda^{\ell}}(\sigma_{\varphi}(g,\mathcal{R}),\mathcal{P}) &= \lambda_{i}^{\ell}\big|_{\sigma_{\varphi}(g,\mathcal{R}).\mathcal{P}} \circ \sigma_{\varphi}(g,\mathcal{R})\big|_{\mathcal{P}} \circ \lambda_{i}^{\ell}\big|_{\mathcal{P}}^{-1} \\ &= \lambda_{i}^{\ell}\big|_{\sigma_{\varphi}(g,\mathcal{R}).\mathcal{P}} \circ \left(\varphi_{\ell}\big|_{g.\mathcal{R}} \circ g\big|_{\mathcal{R}} \circ \varphi_{\ell}\big|_{\mathcal{R}}^{-1}\right)\big|_{\mathcal{P}} \circ \lambda_{i}^{\ell}\big|_{\mathcal{P}}^{-1} \\ &= \lambda_{i}^{\ell}\big|_{\sigma_{\varphi}(g,\mathcal{R}).\mathcal{P}} \circ \varphi_{\ell}\big|_{g.\mathcal{P}'} \circ g\big|_{\mathcal{P}'} \circ \varphi_{\ell}\big|_{\mathcal{P}'}^{-1} \circ \lambda_{i}^{\ell}\big|_{\mathcal{P}}^{-1} \\ &= \lambda_{i}'\big|_{g.\mathcal{P}'} \circ g\big|_{\mathcal{P}'} \circ \lambda_{i}'\big|_{\mathcal{P}'}^{-1} \\ &= \sigma_{\lambda'}(g,\mathcal{P}'). \end{split}$$
(*)



Since $g \in \mathcal{U}_{\Delta}(\mathbf{F})$, the result of Eq. (*) is a permutation in F_i , so we conclude that $\sigma_{\varphi}(g, \mathcal{R}) \in \mathcal{U}_{\Delta_{\ell}}(\mathbf{F}_{\ell})$. Since this holds for every ℓ -panel \mathcal{R} of Φ , this shows that $\iota(g) \in \mathcal{U}_{\Phi}(\mathbf{F}')$.

Conversely, let $g \in \mathcal{U}_{\Phi}(\mathbf{F}')$. We can identify $g \in \operatorname{Aut}(\Phi)$ with a permutation of Δ , and we first verify that this permutation is type-preserving, i.e., that g is in fact an automorphism of Δ . Indeed, let $c \sim_i d$ be *i*-adjacent chambers in Δ . Let $\ell := \ell(i)$ and let \mathcal{R} be the residue of Δ of type I_{ℓ} containing c and d. Then \mathcal{R} is an ℓ -panel of Φ . The local action $\sigma_{\varphi}(g, \mathcal{R})$ is an element of $F'_{\ell} = \mathcal{U}_{\Delta_{\ell}}(\mathbf{F}_{\ell}) \leq \operatorname{Aut}(\Delta_{\ell})$. Hence

$$g\big|_{\mathcal{R}} = \varphi_{\ell}\big|_{g.\mathcal{R}}^{-1} \circ \sigma_{\varphi}(g,\mathcal{R}) \circ \varphi_{\ell}\big|_{\mathcal{R}}$$

is a composition of isomorphisms $\mathcal{R} \to \Delta_{\ell} \to \Delta_{\ell} \to g.\mathcal{R}$, each of which preserves *i*-adjacency. In particular, $g.c \sim_i g.d$. Since c and d were arbitrary, we conclude that $g \in \operatorname{Aut}(\Delta)$.

Next, let \mathcal{P}' be any *i*-panel in Δ with $i \in I_{\ell}$, let \mathcal{R} be the I_{ℓ} -residue of Δ containing \mathcal{P}' and let $\mathcal{P} := \varphi_{\ell}(\mathcal{P}')$ in Δ_{ℓ} . The reverse calculation of Eq. (*) shows that the local action satisfies

$$\sigma_{\lambda'}(q, \mathcal{P}') = \sigma_{\lambda^{\ell}}(\sigma_{\omega}(q, \mathcal{R}), \mathcal{P}).$$

Since $\sigma_{\varphi}(g, \mathcal{R}) \in \mathcal{U}_{\Delta_{\ell}}(F_{\ell})$, we have $\sigma_{\lambda'}(g, \mathcal{P}') \in (F_{\ell})_i = F_i$. We conclude that indeed $g \in \mathcal{U}_{\Delta}(F)$.

In conclusion, the restriction of ι : Aut $(\Delta) \hookrightarrow$ Aut (Φ) to $\mathcal{U}_{\Delta}(\mathbf{F}) \leq$ Aut (Δ) is an isomorphism κ : $\mathcal{U}_{\Delta}(\mathbf{F}) \to \mathcal{U}_{\Phi}(\mathbf{F}')$. Finally, notice that it is obvious that ι is a *homeomorphism* onto its image because Δ and Φ have the same underlying set, and the topology on Aut (Δ) and Aut (Φ) is independent of the additional building structure on Δ and Φ . In particular, κ is an isomorphism of topological groups. \Box CITY PRODUCTS OF RIGHT-ANGLED BUILDINGS AND THEIR UNIVERSAL GROUPS 17

5. Application: Different right-angled buildings of the same type WITH ISOMORPHIC UNIVERSAL GROUPS

Inspired by [Beß22], we provide a construction to produce pairs of right-angled buildings of the same type M over I but with different parameters $(q_i)_{i \in I}$, that nevertheless admit isomorphic universal groups for appropriate choices of the local data \boldsymbol{F} .

The following result is certainly not the most general result possible, but it seems a good trade-off between producing a large amount of examples and still being "readable".

Theorem 5.1. Let M be a right-angled diagram of rank n admitting a non-trivial symmetry $\rho \in \text{Sym}(n)$, and assume that $k \in \{1, \ldots, n\}$ is not fixed by ρ . Let Λ be the support of ρ (i.e., the set of elements not fixed by ρ). For each $\ell \in \{1, \ldots, n\}$, let M_{ℓ} be a right-angled diagram over I_{ℓ} , such that

- $|I_{\ell}| = 1$ for each $\ell \in \Lambda \setminus \{k\}$, and
- $|I_k| =: t > 1$. Write $I_k = \{i_1, \dots, i_t\}$.

Consider the city product $N := \Xi_M(M_1, \ldots, M_n)$, with index set $I = \bigsqcup_{\ell=1}^n I_\ell$.

- For each $i \in I_k$, let G_i and G'_i be two arbitrary permutation groups (acting on sets Ω_i and Ω'_i , respectively).
- For each $i \in I_{\ell}$ for $\ell \notin \{k, \rho(k)\}$, let H_i be an arbitrary permutation group (acting on a set Ω_i).

Finally, consider the two collections F and F' of local data over I defined by

- $F_i := G_i \text{ for } i \in I_k$,
- $F_i := \mathcal{U}_{M_k}(G'_{i_1}, \dots, G'_{i_t})$ for the unique $i \in I_{\rho(k)}$,
- $F_i := H_i$ for all other $i \in I$,
- $F'_i := G'_i$ for $i \in I_k$,
- $F'_i := \mathcal{U}_{M_k}(G_{i_1}, \dots, G_{i_\ell})$ for the unique $i \in I_{\rho^{-1}(k)}$, $F'_i := H_j$ for all $i \in I_\ell$ with $\ell \in \Lambda \setminus \{k, \rho^{-1}(k)\}$, where $I_{\rho(\ell)} = \{j\}$, $F'_i := H_i$ for all other $i \in I$.

Then the universal groups $\mathcal{U}_N(\mathbf{F})$ and $\mathcal{U}_N(\mathbf{F}')$ are isomorphic (as topological groups).

Proof. We will use the notation from Remark 2.25(ii). Consider the collections Land L' of local data over $\{1, \ldots, n\}$ defined by

$$\begin{split} L_k &= \mathcal{U}_{M_k}(G_{i_1}, \dots, G_{i_t}), \\ L_{\rho(k)} &= \mathcal{U}_{M_k}(G'_{i_1}, \dots, G'_{i_t}), \\ L_\ell &= H_i \quad \text{for all } \ell \in \Lambda \setminus \{k, \rho(k)\}, \text{ where } I_\ell = \{i\}, \\ L_\ell &= \mathcal{U}_{M_\ell}(H_i \mid i \in I_\ell) \quad \text{for all } \ell \notin \Lambda, \\ L'_k &= \mathcal{U}_{M_k}(G'_{i_1}, \dots, G'_{i_t}), \\ L'_{\rho^{-1}(k)} &= \mathcal{U}_{M_k}(G_{i_1}, \dots, G_{i_t}), \\ L'_\ell &= H_j \quad \text{for all } \ell \in \Lambda \setminus \{k, \rho^{-1}(k)\}, \text{ where } I_{\rho(\ell)} = \{j\} \\ L'_\ell &= \mathcal{U}_{M_\ell}(H_i \mid i \in I_\ell) \quad \text{for all } \ell \notin \Lambda. \end{split}$$

By Theorem 4.1, we then have

$$\mathcal{U}_N(F) \cong \mathcal{U}_M(L)$$
 and
 $\mathcal{U}_N(F') \cong \mathcal{U}_M(L').$

Since $L'_{\ell} = L_{\rho(\ell)}$ for all ℓ , it is now immediately clear that the symmetry ρ of the diagram M ensures that $\mathcal{U}_M(\mathbf{L}) \cong \mathcal{U}_M(\mathbf{L}')$, and the result follows. \square **Examples 5.2.** (1) The first set of examples covers precisely the cases that can also be obtained with Lara Beßman's method from [Beß22].

Let M be the diagram of rank 2 with label ∞ , let M_1 be the diagram of rank 1 and let M_2 be the diagram of rank t > 1 without edges. The city product $N := \Xi_M(M_1, M_2)$ of these diagrams has rank t + 1; we label the t + 1 nodes of N as below.



We can now apply Theorem 5.1, with ρ the unique non-trivial symmetry of Mand with k = 2, so $\rho(k) = \rho^{-1}(k) = 1$ and $|I_k| = t > 1$. For each $i \in \{1, \ldots, t\}$, let G_i and G'_i be two arbitrary permutation groups (acting on sets Ω_i and Ω'_i , respectively). Notice that there are no values $\ell \notin \{k, \rho(k)\}$, so we do not have to choose groups H_i as in the theorem. Moreover, notice that because M_2 has no edges, we simply have $\mathcal{U}_{M_2}(G_1, \ldots, G_t) \cong G_1 \times \cdots \times G_t$ (see Lemma 2.26). It now follows from Theorem 5.1 that

 $\mathcal{U}_N(G_1, G_2, \ldots, G_t, G'_1 \times \cdots \times G'_t) \cong \mathcal{U}_N(G'_1, G'_2, \ldots, G'_t, G_1 \times \cdots \times G_t).$

(2) We now present an example with a non-involutory symmetry. Let M be the diagram of rank 3 with all labels ∞ , let M_1 and M_3 be diagrams of rank 1 and let M_2 be the diagram of rank 2 without edges. The city product $N := \sum_M (M_1, M_2, M_3)$ of these diagrams has rank 4, labeled as below.



We now choose ρ to be the cyclic symmetry (123) of M of order 3 and we choose k = 2, so $\rho(k) = 3$ and $\rho^{-1}(k) = 1$. We let G_1, G_2, G'_1, G'_2, H be five permutation groups (acting on sets $\Omega_1, \Omega_2, \Omega'_1, \Omega'_2, \Omega$, respectively).

It now follows from Theorem 5.1 that

 $\mathcal{U}_N(G_1, G_2, G'_1 \times G'_2, H) \cong \mathcal{U}_N(G'_1, G'_2, H, G_1 \times G_2).$

Notice that both examples can exist for *locally finite* buildings (i.e., buildings where each of the parameters q_i , $i \in I$ is finite). This happens because the diagram M_k has no edges, so that the corresponding universal group $\mathcal{U}_{M_k}(G_1,\ldots,G_t)$ is just a direct product of the groups G_i . On the other hand, if the diagram M_k has edges, then the universal group $\mathcal{U}_{M_k}(G_1,\ldots,G_t)$ is never finite, so those examples do not occur in the locally finite case.

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Email address, Tom De Medts (corresponding author): tom.demedts@ugent.be

(Address for both authors) Ghent University, Department of Mathematics: Algebra and Geometry, Krijgslaan 281, S25, 9000 Ghent, Belgium