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Phase and group velocities for shear wave propagation in an incompressible, hyperelastic material with uniaxial stretch

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Abstract. Objective. Determining elastic properties of materials from observations of shear wave propagation is difficult in anisotropic materials because of the complex relations among the propagation direction, shear wave polarizations, and material symmetries. In this study, we derive expressions for the phase velocities of the SH and SV propagation modes as a function of propagation direction in an incompressible, hyperelastic material with uniaxial stretch. Approach. Wave motion is included in the material model by adding incremental, small amplitude motion to the initial, finite deformation. Equations of motion for the SH and SV propagation modes are constructed using the Cauchy stress tensor derived from the strain energy function of the material. Group velocities for the SH and SV propagation modes are derived from the angle-dependent phase velocities. Main results. Sample results are presented for the Arruda-Boyce, Mooney-Rivlin, and Isihara material models using model parameters previously determined in a phantom. Significance. Results for the Mooney-Rivlin and Isihara models demonstrate shear splitting in which the SH and SV propagation modes have unequal group velocities for propagation across the material symmetry axis. In addition, for sufficiently large stretch, the Arruda-Boyce and Isihara material models show cusp structures with triple-valued group velocities for the SV mode at angles of roughly 15° to the material symmetry axis.

Keywords: elastography, shear wave, phase velocity, group velocity, hyperelastic material, uniaxial stretch

1. Introduction

In a recent study, Caenen *et al* (2020) reported observations of shear wave propagation following an acoustic radiation force impulse (ARFI) excitation in a stretched, polyvinyl alcohol (PVA) phantom using a unique experimental configuration in which the stretch axis of the phantom was tilted with respect to the plane perpendicular to the excitation axis, see figure 2 of Caenen *et al* (2020). The process of stretching the (initially isotropic) PVA introduces a symmetry axis in the material, making it a transversely isotropic (TI) material with rotational symmetry about the axis, and reflection symmetry across any plane parallel to the axis. This material can support both SH and SV shear wave propagation modes with propagation velocities that depend on the material stretch. The tilted excitation configuration used by Caenen *et al* (2020) allowed both the SH and SV propagation modes to be excited simultaneously, and thereby allowed observation of features such as shear splitting that cannot be observed in the common, nontilted experimental setup in which only the SH propagation mode is observed.

Caenen *et al* (2020) analyzed their measurements by performing mechanical testing in a sample of the PVA phantom to measure the stress-strain relation of the material, which was then used to determine model coefficients for three hyperelastic models, the Arruda-Boyce (1993), Mooney-Rivlin (Mooney 1940, Rivlin and Saunders 1951), and Isihara (1951) models. These model coefficients were used in finite element simulations with both tilted and non-tilted excitation configurations to generate shear wave signals for stretch values ranging from zero to 114% as realized in the experiment. Results of the simulations agreed with the observed measurements and, in particular, showed the presence of shear wave splitting with the tilted excitation configuration for larger values of stretch, thereby confirming the observation of both SH and SV wave propagation in the material.

The variation in wave speed in a hyperelastic material with static deformation is known as the acoustoelastic effect. Previous studies of this effect have been performed in compressed and stretched phantoms including the studies by Gennisson $et \ al \ (2007)$, Urban et al (2015), and Chatelin et al (2014). In particular, Gennisson et al (2007) observed both the SH and SV polarization modes for shear wave propagation in orthogonal directions relative to the material axis of a compressed PVA phantom, see figure 1 of Gennisson et al (2007). Rosen and Jiang (2019) have analyzed this experimental geometry in the context of hyperelastic material models to relate the measured speeds to material properties. The investigations reported by Urban et al (2015) and Chatelin et al (2014) observed shear wave propagation with an experimental configuration where the SH wave mode was observed for a full range of propagation directions in a plane oriented perpendicular to the excitation axis, see, for example, figure 2 of Chatelin *et al* (2014). These measurements were analyzed by fitting an elliptical shape to the measured speeds as a function of the propagation direction and determining the shear moduli for propagation along and across the material symmetry axis. In addition, Gennisson *et al* (2007) and Urban *et al* (2015) determined the

 nonlinear shear modulus of the material by analyzing the measured shear moduli as a function of the material deformation using a third order expansion of the strain energy (Hamilton *et al* 2004, Destrade *et al* 2010).

The measurements reported by Caenen *et al* (2020) extend these investigations by introducing a tilted excitation axis to allow the simultaneous excitation and observation of SH and SV propagation modes over a full range of propagation directions. However, analysis of these measurements is more complicated because the experiments observe the group propagation velocities which, in general, are equal to the corresponding phase velocities only for propagation in orthogonal directions relative to the material symmetry axis. More general expressions for the SH and SV group velocities require angular-dependent expressions for the phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ for arbitrary propagation direction. The phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ are also required in Green's tensor calculations of shear wave signals as described by Rouze *et al* (2020).

In this study, we derive analytic expressions for the phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ of small amplitude wave propagation as a function of propagation direction θ relative to the symmetry axis of an incompressible, hyperelastic material with finite, uniaxial stretch. This derivation follows the procedure described by Boulanger and Hayes (1992, 2001) by modeling a material with finite stretch using the perspective of nonlinear solid mechanics (Holzapfel 2000), and then adding incremental, small amplitude wave motion. Corresponding group propagation velocities can be derived from $v_{SH}(\theta)$ and $v_{SV}(\theta)$ for comparison with experimental measurements. Sample results are presented for the phase and group velocities in the Arruda-Boyce, Mooney-Rivlin, and Isihara material models using model coefficients determined by Caenen *et al* (2020). These results demonstrate the complex structures in angular-dependent observations of SH and SV wave propagation and suggest methods for measuring material model parameters from these observations.

2. Derivation of the phase and group velocities

2.1. Finite, uniaxial stretch

Figure 1 shows the coordinate system used to describe wave propagation in an isotropic hyperelastic material with finite, uniaxial stretch. As sketched in figures 1(a) and 1(b), the stretch causes a point P at reference position $\mathbf{X} = (X, Y, Z)^T = (X_1, X_2, X_3)^T$ in the undeformed material to move to position $\mathbf{x} = (x, y, z)^T = (x_1, x_2, x_3)^T$ in the deformed material. This stretch introduces a symmetry axis \hat{A} , and the material is transversely isotropic (TI) with rotational symmetry about \hat{A} , and reflection symmetry across any plane containing \hat{A} . For an incompressible material with stretch λ in the \hat{z} direction, the reference and deformed positions are related as

$$x = \frac{X}{\sqrt{\lambda}}, \qquad y = \frac{Y}{\sqrt{\lambda}}, \qquad \text{and} \qquad z = \lambda Z.$$
 (1)

The deformation gradient tensor **F** with components $F_{iA} = \partial x_i / \partial X_A$ and the left



Figure 1. Coordinate system used for the analysis of wave propagation in a stretched, hyperelastic material. Figure 1(a) shows an isotropic, hyperelastic material in the undeformed (reference) state with stretch $\lambda = 1$ and a point P at the coordinates (X, Y, Z). Figure 1(b) shows the material after a uniaxial stretch in the z direction with $\lambda = 1.8$ so that point P is located at the coordinates (x, y, z) given by (1). Figure 1(c) shows the coordinates used to analyze wave propagation in the stretched material. The material symmetry axis \hat{A} is aligned with the z axis, and shear wave propagation is observed in the x - z plane with a propagation direction \hat{n} at an angle θ relative to the z axis. Polarization vectors for the P, SH, and SV propagation modes are defined relative to the $\hat{A} - \hat{n}$ plane with the SH polarization vector oriented perpendicular to \hat{n} , and the P polarization vector oriented in the \hat{n} direction. For this analysis, wave motion in the P propagation mode is ignored because the material is assumed to be incompressible. The ζ, ξ, η coordinate system is aligned with the polarization vectors.

Cauchy-Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are given by

$$\mathbf{F} = \begin{pmatrix} 1/\sqrt{\lambda} & 0 & 0\\ 0 & 1/\sqrt{\lambda} & 0\\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1/\lambda & 0 & 0\\ 0 & 1/\lambda & 0\\ 0 & 0 & \lambda^2 \end{pmatrix}. \quad (2)$$

Tensor invariants I_1 , I_2 , and I_3 of **B** are given by

$$I_1 = \operatorname{tr} \mathbf{B} = \lambda^2 + \frac{2}{\lambda}, \quad I_2 = \frac{1}{2} \left[(\operatorname{tr} \mathbf{B})^2 - \operatorname{tr} \left(\mathbf{B}^2 \right) \right] = 2\lambda + \frac{1}{\lambda^2}, \quad \text{and} \quad I_3 = \det \mathbf{B} = 1 \quad (3)$$

where $I_3 = 1$ because the material is incompressible.

2.2. Shear wave propagation superimposed on the static deformation

Following Boulanger and Hayes (1992, 2001), we model wave motion in the stretched material by adding wave displacements to the initial deformation (1). Figure 1(c) shows the coordinate system used to describe plane wave motion with propagation direction \hat{n} in the x - z plane at an angle θ relative to the $\hat{z} = \hat{A}$ axis. Wave motion is described in terms of the P, SH, and SV propagation modes identified by their polarization relative to the propagation direction \hat{n} and material symmetry axis \hat{A} (Tsvankin 2012, Carcione 2015, Rouze *et al* 2013). The P propagation mode has longitudinal polarization and corresponds to the acoustic wave. The SH and SV propagation modes correspond to shear wave motion with transverse polarization perpendicular to the to the $\hat{A} - \hat{n}$ plane

for the SH mode, and transverse polarization in the $\widehat{A} - \widehat{n}$ plane for the SV mode. Then, the propagation direction and polarization vectors are given by

$$\widehat{n} = \widehat{P}^P = \widehat{\eta} = \begin{pmatrix} \sin\theta\\0\\\cos\theta \end{pmatrix}, \quad \widehat{P}^{SH} = \widehat{\xi} = \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \quad \text{and} \quad \widehat{P}^{SV} = \widehat{\zeta} = \begin{pmatrix} \cos\theta\\0\\-\sin\theta \end{pmatrix}$$
(4)

where $\hat{\zeta}$, $\hat{\xi}$, and $\hat{\eta}$ are unit vectors for the ζ , ξ , η coordinate system aligned with the polarization axes. For the incompressible material considered here, the speed of the acoustic (P-mode) wave diverges, and only the SH and SV propagation modes are considered henceforth.

We use a tilde notation to describe the combination of wave motion superimposed on the stretched material. The total material deformation is given by

$$\widetilde{\mathbf{x}} = \mathbf{x} + \widehat{P}^{SH} f(\mathcal{A}_{\rm SH}, \eta, t) + \widehat{P}^{SV} g(\mathcal{A}_{\rm SV}, \eta, t)$$
(5)

where **x** is the deformation (1) without wave motion, and the two additional terms describe plane wave motion with polarizations \hat{P}^{SH} and \hat{P}^{SV} for the SH and SV propagation modes, respectively. Wave motion in the P propagation mode is not included in this expression because the material is assumed to be incompressible. The wavefunctions $f(\mathcal{A}_{\rm SH}, \eta, t)$ and $g(\mathcal{A}_{\rm SV}, \eta, t)$ can be written in terms of the coordinate $\eta = \hat{n} \cdot \vec{x}$ along the propagation direction as

$$f(\mathcal{A}_{\rm SH},\eta,t) = \mathcal{A}_{\rm SH}e^{ik(\eta-v_{\rm SH}t)} \quad \text{and} \quad g(\mathcal{A}_{\rm SV},\eta,t) = \mathcal{A}_{\rm SV}e^{ik(\eta-v_{\rm SV}t)} \tag{6}$$

where \mathcal{A}_{SH} and \mathcal{A}_{SV} are the amplitudes, and v_{SH} and v_{SV} are the phase velocities of the SH and SV propagation modes, respectively.

Including wave motion, components of the total deformation gradient tensor $\widetilde{\mathbf{F}}$ can be calculated as $\widetilde{F}_{iA} = \partial \widetilde{x}_i / \partial X_A$. To simplify this calculation, we introduce a tensor **M** with components $M_{ij} = \partial \widetilde{x}_i / \partial x_j$ that characterizes the incremental deformation of the superimposed wave motion. Then, components \widetilde{F}_{iA} are given by

$$\widetilde{F}_{iA} = \frac{\partial \widetilde{x}_i}{\partial X_A} = \frac{\partial \widetilde{x}_i}{\partial X_j} \frac{\partial x_j}{\partial X_A} = M_{ij} F_{jA}$$
(7)

and $\widetilde{\mathbf{F}}$ is given by the product $\widetilde{\mathbf{F}} = \mathbf{MF}$. Components M_{ij} of \mathbf{M} can be calculated from (5) and (6) using the relation $\eta = \widehat{n} \cdot \vec{x} = \widehat{n}_k x_k$,

$$M_{ij} = \frac{\partial \widetilde{x}_i}{\partial x_j} = \delta_{ij} + \widehat{P}_i^{\rm SH} \frac{\partial f}{\partial \eta} \,\widehat{n}_j + \widehat{P}_i^{\rm SV} \frac{\partial g}{\partial \eta} \,\widehat{n}_j \tag{8}$$

Then, using the notation $f_{\eta} = \partial f / \partial \eta$, $g_{\eta} = \partial g / \partial \eta$, $s = \sin \theta$, and $c = \cos \theta$, the incremental deformation tensor **M** can be written as

$$\mathbf{M} = \begin{pmatrix} 1 + g_{\eta} sc & 0 & g_{\eta} c^{2} \\ f_{\eta} s & 1 & f_{\eta} c \\ -g_{\eta} s^{2} & 0 & 1 - g_{\eta} sc \end{pmatrix}.$$
 (9)

The total deformation tensor $\widetilde{\mathbf{B}} = \widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^T = \mathbf{M}\mathbf{B}\mathbf{M}^T$ is given by The total deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{T} = \mathbf{M}\mathbf{B}\mathbf{M}^{T}$ is given by $\begin{aligned}
& \begin{pmatrix} (1+g_{\eta}sc)^{2}/\lambda & f_{\eta}s\left(1+g_{\eta}sc\right)/\lambda & -g_{\eta}s^{2}\left(1+g_{\eta}sc\right)/\lambda \\
& +\lambda^{2}g_{\eta}c^{2}c^{4} & +\lambda^{2}f_{\eta}g_{\eta}c^{3} & +\lambda^{2}g_{\eta}c^{2}\left(1-g_{\eta}sc\right) \\
& f_{\eta}s\left(1+g_{\eta}sc\right)/\lambda & (f_{\eta}^{2}s^{2}+1)/\lambda & -f_{\eta}g_{\eta}s^{3}/\lambda \\
& +\lambda^{2}f_{\eta}g_{\eta}c^{3} & +\lambda^{2}f_{\eta}^{2}c^{2} & +\lambda^{2}f_{\eta}c\left(1-g_{\eta}sc\right) \\
& -g_{\eta}s^{2}\left(1+g_{\eta}sc\right)/\lambda & -f_{\eta}g_{\eta}s^{3}/\lambda & g_{\eta}^{2}s^{4}/\lambda \\
& +\lambda^{2}g_{\eta}c^{2}\left(1-g_{\eta}sc\right) & +\lambda^{2}f_{\eta}c\left(1-g_{\eta}sc\right) & +\lambda^{2}\left(1-g_{\eta}sc\right)^{2}
\end{aligned}$ In addition, \mathbf{M}^{-1} is given by $\mathbf{M}^{-1} = \begin{pmatrix} 1-g_{\eta}sc & 0 & -g_{\eta}c^{2} \\
& -f_{\eta}s & 1 & -f_{\eta}c \\
& g_{\eta}s^{2} & 0 & 1+g_{\eta}sc
\end{aligned}$ (11)

and
$$\widetilde{\mathbf{B}}^{-1} = \left(\mathbf{M}\mathbf{B}\mathbf{M}^{T}\right)^{-1} = \left(\mathbf{M}^{T}\right)^{-1}\mathbf{B}^{-1}\mathbf{M}^{-1}$$
 is given by

$$\widetilde{\mathbf{B}}^{-1} = \begin{pmatrix} \lambda (1 - g_{\eta} sc)^{2} & -\lambda f_{\eta} s & \lambda \left[\left(g_{\eta}^{2} c^{2} + f_{\eta}^{2} \right) sc - g_{\eta} c^{2} \right] \\ + \lambda f_{\eta}^{2} s^{2} + g_{\eta}^{2} s^{4} / \lambda^{2} & -\lambda f_{\eta} s & \lambda \left[\left(g_{\eta}^{2} c^{2} + f_{\eta}^{2} \right) sc - g_{\eta} c^{2} \right] \\ - \lambda f_{\eta} s & \lambda & -\lambda f_{\eta} c \\ \lambda \left[\left(g_{\eta}^{2} c^{2} + f_{\eta}^{2} \right) sc - g_{\eta} c^{2} \right] & -\lambda f_{\eta} c & \lambda \left(g_{\eta}^{2} c^{2} + f_{\eta}^{2} \right) c^{2} \\ + g_{\eta} s^{2} \left(1 + g_{\eta} sc \right) / \lambda^{2} & -\lambda f_{\eta} c & \lambda \left(g_{\eta}^{2} c^{2} + f_{\eta}^{2} \right) c^{2} \\ + \left(1 + g_{\eta} sc \right)^{2} / \lambda^{2} & -\lambda f_{\eta} c & + \left(1 + g_{\eta} sc \right)^{2} / \lambda^{2} \end{pmatrix}.$$
(12)

 $\begin{pmatrix} +g_{\eta}s^{2} (1+g_{\eta}sc)/\lambda^{2} \\ +(1+g_{\eta}sc)/\lambda^{2} \end{pmatrix} + (1+g_{\eta}sc)/\lambda^{2} \end{pmatrix}$ Invariants $\widetilde{I}_{1}, \widetilde{I}_{2}$, and \widetilde{I}_{3} of $\widetilde{\mathbf{B}}$ can be calculated using (10) and (12). For \widetilde{I}_{1} ,

$$\widetilde{I}_{1} = \operatorname{tr} \widetilde{\mathbf{B}} = \lambda^{2} + \frac{2}{\lambda} - 2g_{\eta} \left(\lambda^{2} - \frac{1}{\lambda}\right) sc + \left(f_{\eta}^{2} + g_{\eta}^{2}\right) \left(\lambda^{2}c^{2} + \frac{1}{\lambda}s^{2}\right).$$

$$(13)$$

Invariant I_3 is given by

$$\widetilde{I}_{3} = \det \widetilde{\mathbf{B}} = \det \left(\mathbf{M} \mathbf{B} \mathbf{M}^{T} \right) = (\det \mathbf{M}) \left(\det \mathbf{B} \right) \left(\det \mathbf{M}^{T} \right) = 1.$$
(14)

As expected, the result $I_3 = 1$ indicates that the material with added wave motion is incompressible because only shear (isovolumetric) wave motion is included in (5). Then, using (14), invariant I_2 is given by Holzapfel (2000),

$$\widetilde{I}_2 = \det \widetilde{\mathbf{B}} \operatorname{tr} \widetilde{\mathbf{B}}^{-1} = 2\lambda + \frac{1}{\lambda^2} - 2g_\eta \left(\lambda - \frac{1}{\lambda^2}\right) sc + \lambda f_\eta^2 + g_\eta^2 \left(\lambda c^2 + \frac{1}{\lambda^2} s^2\right).$$
(15)

2.3. Cauchy stress tensor and the equations of motion

Isotropic, hyperelastic materials can be characterized by a strain energy function Wwhich is a function of the invariants I_1 , I_2 , and I_3 of **B** (Holzapfel 2000). For the incompressible material considered here, $I_3 = 1$. In addition, from (14), $I_3 = 1$, and the strain energy W for the stretched material with superimposed wave motion can be written as a function of the invariants \tilde{I}_1 and \tilde{I}_2 as $\tilde{W} = W\left(\tilde{I}_1, \tilde{I}_2\right)$. The Cauchy stress tensor $\tilde{\sigma}$ is determined from the strain energy \tilde{W} by the relation (Holzapfel 2000)

$$\widetilde{\boldsymbol{\sigma}} = -p\mathbf{1} + 2\frac{\partial \widetilde{W}}{\partial \widetilde{I}_1}\widetilde{\mathbf{B}} - 2\frac{\partial \widetilde{W}}{\partial \widetilde{I}_2}\widetilde{\mathbf{B}}^{-1}$$
(16)

where p is a Lagrange multiplier in the form of a hydrostatic pressure that is required to enforce the incompressibility of the material.

The equation of motion for a point at position $\tilde{\mathbf{x}}$ can be written using the divergence of $\tilde{\boldsymbol{\sigma}}$ calculated with respect to the x_1, x_2, x_3 coordinates,

$$\rho \, \frac{\partial^2 \widetilde{x}_i}{\partial t^2} = \frac{\partial \widetilde{\sigma}_{ij}}{\partial x_j}.$$

However, because $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{B}}^{-1}$ are functions of the position $\eta = \widehat{n} \cdot \vec{x}$ along the propagation direction, it is easier to evaluate the divergence of $\widetilde{\boldsymbol{\sigma}}$ using the ζ, ξ, η coordinate system shown in figure 1(c) that is aligned with the propagation and polarization directions. Then, using (5), the $\widetilde{\eta}, \widetilde{\xi}, \widetilde{\zeta}$ coordinates of a point in the deformed material, including wave motion, are given by

$$\widetilde{\eta} = \eta, \qquad \widetilde{\xi} = \xi + f(\eta, t), \quad \text{and} \quad \widetilde{\zeta} = \zeta + g(\eta, t)$$
(18)

(17)

where the equality $\eta = \tilde{\eta}$ holds because the material is incompressible and longitudinal wave motion is ignored.

The equation of motion for the SH wavefunction $f(\mathcal{A}_{\rm SH}, \eta, t)$ can be derived by evaluating (17) for the $\tilde{\xi}$ coordinate,

$$\rho \,\frac{\partial^2 \widetilde{\xi}}{\partial t^2} = \frac{\partial \widetilde{\sigma}_{\xi\xi}}{\partial \zeta} + \frac{\partial \widetilde{\sigma}_{\xi\xi}}{\partial \xi} + \frac{\partial \widetilde{\sigma}_{\xi\eta}}{\partial \eta}.\tag{19}$$

In this expression, $\partial \tilde{\sigma}_{\xi\zeta} / \partial \zeta = \partial \tilde{\sigma}_{\xi\xi} / \partial \xi = 0$ because $\tilde{\sigma}$ is only a function of the coordinate η through the dependence on the wavefunctions $f(\mathcal{A}_{\rm SH}, \eta, t)$ and $g(\mathcal{A}_{\rm SV}, \eta, t)$ in (6), (10), and (12). Also, the only time dependence in $\tilde{\xi}$ is through its dependence on the wavefunction $f(\mathcal{A}_{\rm SH}, \eta, t)$ in (18). Then, using (16), the equation of motion for the SH wavefunction $f(\mathcal{A}_{\rm SH}, \eta, t)$ is given by

$$\rho \frac{\partial^2 f}{\partial t^2} = \frac{\partial \widetilde{\sigma}_{\xi\eta}}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\widehat{\xi} \cdot \widetilde{\boldsymbol{\sigma}} \, \widehat{\eta} \right) = \frac{\partial}{\partial \eta} \left(2 \frac{\partial \widetilde{W}}{\partial \widetilde{I}_1} \, \widehat{\xi} \cdot \widetilde{\mathbf{B}} \, \widehat{\eta} - 2 \frac{\partial \widetilde{W}}{\partial \widetilde{I}_2} \, \widehat{\xi} \cdot \widetilde{\mathbf{B}}^{-1} \, \widehat{\eta} \right) \tag{20}$$

where the term involving the pressure p vanishes because $\hat{\xi} \cdot \mathbf{1} \hat{\eta} = 0$. Similarly, the equation of motion for the SV wavefunction $g(\mathcal{A}_{sv}, \eta, t)$ is given by

$$\rho \frac{\partial^2 g}{\partial t^2} = \frac{\partial \widetilde{\sigma}_{\zeta \eta}}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\widehat{\zeta} \cdot \widetilde{\boldsymbol{\sigma}} \, \widehat{\eta} \right) = \frac{\partial}{\partial \eta} \left(2 \frac{\partial \widetilde{W}}{\partial \widetilde{I}_1} \, \widehat{\zeta} \cdot \widetilde{\mathbf{B}} \, \widehat{\eta} - 2 \frac{\partial \widetilde{W}}{\partial \widetilde{I}_2} \, \widehat{\zeta} \cdot \widetilde{\mathbf{B}}^{-1} \, \widehat{\eta} \right). \tag{21}$$

In (20) and (21), components $\hat{\xi} \cdot \tilde{\mathbf{B}} \hat{\eta}$, $\hat{\zeta} \cdot \tilde{\mathbf{B}} \hat{\eta}$, $\hat{\xi} \cdot \tilde{\mathbf{B}}^{-1} \hat{\eta}$, and $\hat{\zeta} \cdot \tilde{\mathbf{B}}^{-1} \hat{\eta}$ can be evaluated using (4), (10), and (12),

$$\widehat{\boldsymbol{\xi}} \cdot \widetilde{\mathbf{B}} \,\widehat{\boldsymbol{\eta}} = f_{\eta} \left(\lambda^2 c^2 + \frac{1}{\lambda} s^2 \right), \qquad \widehat{\boldsymbol{\zeta}} \cdot \widetilde{\mathbf{B}} \,\widehat{\boldsymbol{\eta}} = -\left(\lambda^2 - \frac{1}{\lambda} \right) sc + g_{\eta} \left(\lambda^2 c^2 + \frac{1}{\lambda} s^2 \right),$$
$$\widehat{\boldsymbol{\xi}} \cdot \widetilde{\mathbf{B}}^{-1} \,\widehat{\boldsymbol{\eta}} = -f_{\eta} \lambda, \qquad \text{and} \qquad \widehat{\boldsymbol{\zeta}} \cdot \widetilde{\mathbf{B}}^{-1} \,\widehat{\boldsymbol{\eta}} = \left(\lambda - \frac{1}{\lambda^2} \right) sc - g_{\eta} \left(\lambda c^2 + \frac{1}{\lambda^2} s^2 \right). \tag{22}$$

2.4. Small amplitude wave motion

The equations of motion (20) and (21) are complicated functions of the wave amplitudes $\mathcal{A}_{_{\mathrm{SH}}}$ and $\mathcal{A}_{_{\mathrm{SV}}}$ through the dependence of $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{B}}^{-1}$ on the wavefunctions $f(\mathcal{A}_{_{\mathrm{SH}}}, \eta, t)$ and $g(\mathcal{A}_{_{\mathrm{SV}}}, \eta, t)$ in (10) and (12). The equations of motion are first order in $\mathcal{A}_{_{\mathrm{SH}}}$ and $\mathcal{A}_{_{\mathrm{SV}}}$ only for special-case material models such as the Mooney-Rivlin material considered by Boulanger and Hayes (1992, 2001). For more general cases, we consider small amplitude waves and derive equations of motion by expanding the right hand sides of (20) and (21) in a Taylor series in $\mathcal{A}_{_{\mathrm{SH}}}$ and $\mathcal{A}_{_{\mathrm{SV}}}$ and keeping only terms of $O(\mathcal{A}_{_{\mathrm{SH}}})$ and $O(\mathcal{A}_{_{\mathrm{SV}}})$.

First, consider the expressions $\partial \tilde{W}/\partial \tilde{I}_1$ and $\partial \tilde{W}/\partial \tilde{I}_2$ in (20) and (21). Using the subscript 0 to indicate terms evaluated with $\mathcal{A}_{\rm SH} = 0$ and $\mathcal{A}_{\rm SV} = 0$, the partial derivative $\partial \tilde{W}/\partial \tilde{I}_1$ is given by

$$\frac{\partial \widetilde{W}}{\partial \widetilde{I}_{1}} = \left(\frac{\partial \widetilde{W}}{\partial \widetilde{I}_{1}}\right)_{0} + \left(\frac{\partial^{2} \widetilde{W}}{\partial \widetilde{I}_{1}^{2}} \frac{\partial \widetilde{I}_{1}}{\partial f} \frac{\partial f}{\partial \mathcal{A}_{SH}} + \frac{\partial^{2} \widetilde{W}}{\partial \widetilde{I}_{1} \partial \widetilde{I}_{2}} \frac{\partial \widetilde{I}_{2}}{\partial f} \frac{\partial f}{\partial \mathcal{A}_{SH}}\right)_{0} \mathcal{A}_{SH} + \left(\frac{\partial^{2} \widetilde{W}}{\partial \widetilde{I}_{1}^{2}} \frac{\partial \widetilde{I}_{1}}{\partial g} \frac{\partial g}{\partial \mathcal{A}_{SV}} + \frac{\partial^{2} \widetilde{W}}{\partial \widetilde{I}_{1} \partial \widetilde{I}_{2}} \frac{\partial \widetilde{I}_{2}}{\partial g} \frac{\partial g}{\partial \mathcal{A}_{SV}}\right)_{0} \mathcal{A}_{SV} + \cdots$$
(23)

In this expression, partial derivatives such as $\left(\partial \widetilde{W}/\partial \widetilde{I}_{1}\right)_{0}$ are evaluated using the strain energy $W(I_{1}, I_{2})$ without the added wave motion, for example, $\left(\partial \widetilde{W}/\partial \widetilde{I}_{1}\right)_{0} = \partial W/\partial I_{1}$. Also, partial derivatives like $\left(\partial \widetilde{I}_{1}/\partial f\right)_{0}$ can be evaluated using (13) and (15) so that (23) is given by

$$\frac{\partial \widetilde{W}}{\partial \widetilde{I}_1} = \frac{\partial W}{\partial I_1} + \left[-2 \frac{\partial^2 W}{\partial I_1^2} \left(\lambda^2 - \frac{1}{\lambda} \right) - 2 \frac{\partial^2 W}{\partial I_1 \partial I_2} \left(\lambda - \frac{1}{\lambda^2} \right) \right] g_\eta sc + \cdots$$
(24)

Similarly, $\partial W/\partial I_2$ is given by

$$\frac{\partial \widetilde{W}}{\partial \widetilde{I}_2} = \frac{\partial W}{\partial I_2} + \left[-2 \frac{\partial^2 W}{\partial I_1 \partial I_2} \left(\lambda^2 - \frac{1}{\lambda} \right) - 2 \frac{\partial^2 W}{\partial I_2^2} \left(\lambda - \frac{1}{\lambda^2} \right) \right] g_\eta sc + \cdots .$$
(25)
Finally, the equation of motion for the SH wavefunction $f(A_1, n, t)$ can be found

Finally, the equation of motion for the SH wavefunction $f(\mathcal{A}_{\rm SH}, \eta, t)$ can be found by combining (20), (24), (25), and (22). Keeping only terms of $O(\mathcal{A}_{\rm SH})$ and $O(\mathcal{A}_{\rm SV})$ and recalling the notation $s = \sin \theta$ and $c = \cos \theta$ gives

$$\rho \frac{\partial^2 f}{\partial t^2} = \rho v_{\rm SH}^2 \frac{\partial^2 f}{\partial \eta^2} \tag{26}$$

where the angle-dependent SH phase velocity $v_{\rm SH}(\theta)$ is given by

$$\rho v_{\rm SH}^2(\theta) = \left(2\lambda^2 \frac{\partial W}{\partial I_1} + 2\lambda \frac{\partial W}{\partial I_2}\right)\cos^2\theta + \left(\frac{2}{\lambda} \frac{\partial W}{\partial I_1} + 2\lambda \frac{\partial W}{\partial I_2}\right)\sin^2\theta.$$
(27)

Expression (27) can be written as

$$w_{\rm SH}^2(\theta) = \mu_{\parallel} \cos^2 \theta + \mu_{\perp} \sin^2 \theta \tag{28}$$

where μ_{\parallel} and μ_{\perp} are shear moduli that characterize wave propagation for the SH mode in directions parallel and perpendicular to the material symmetry axis,

$$\mu_{\parallel} = 2\lambda^2 \frac{\partial W}{\partial I_1} + 2\lambda \frac{\partial W}{\partial I_2} \quad \text{and} \quad \mu_{\perp} = \frac{2}{\lambda} \frac{\partial W}{\partial I_1} + 2\lambda \frac{\partial W}{\partial I_2}.$$
(29)

We note that (28) has the same form as for wave propagation in a linear, elastic material with an intrinsic TI structure such as given by equation (16) of Rouze *et al* (2013).

Similarly, the equation of motion for the SV wavefunction $g(\mathcal{A}_{SV}, \eta, t)$ can be found by combining (21), (24), (25), and (22) and keeping only terms of $O(\mathcal{A}_{SH})$ and $O(\mathcal{A}_{SV})$,

$$\rho \frac{\partial^2 g}{\partial t^2} = \rho v_{\rm SV}^2 \frac{\partial^2 g}{\partial \eta^2}$$

where the angle-dependent SV phase velocity $v_{\rm SV}(\theta)$ is given by

$$\rho v_{\rm SV}^2(\theta) = \left(2\lambda^2 \frac{\partial W}{\partial I_1} + 2\lambda \frac{\partial W}{\partial I_2}\right) \cos^2 \theta + \left(\frac{2}{\lambda} \frac{\partial W}{\partial I_1} + \frac{2}{\lambda^2} \frac{\partial W}{\partial I_2}\right) \sin^2 \theta + \left[4\frac{\partial^2 W}{\partial I_1^2} \left(\lambda^2 - \frac{1}{\lambda}\right)^2 + 4\frac{\partial^2 W}{\partial I_2^2} \left(\lambda - \frac{1}{\lambda^2}\right)^2 + 8\frac{\partial^2 W}{\partial I_1 \partial I_2} \left(\lambda^2 - \frac{1}{\lambda}\right) \left(\lambda - \frac{1}{\lambda^2}\right)\right] \sin^2 \theta \cos^2 \theta.$$
(31)

(30)

We note that this result is equivalent to the special-case result from Ogden (2007) for the velocity of small amplitude plane waves as a function of propagation direction in an incompressible, pre-stressed material. However, this equivalence may not be apparent because of the different notations used. A demonstration of the equivalence of the result from Ogden (2007) and the result (31) is given in supplemental data associated with this paper.

For the special case of an unstretched, isotropic material with $\lambda = 1$, (27) and (31) indicate that ρv_{SH}^2 and ρv_{SV}^2 are independent of the propagation angle θ and equal to the shear modulus μ of the material,

$$\mu = \rho v_{SH}^2(\lambda = 1) = \rho v_{SV}^2(\lambda = 1) = 2 \left. \frac{\partial W}{\partial I_1} \right|_{\lambda = 1} + 2 \left. \frac{\partial W}{\partial I_2} \right|_{\lambda = 1}.$$
(32)

Finally, we note that an important result from the derivation of the equations of motion (26) and (30) from (20) and (21) is that the resulting equations reduce to separate relations for the wavefunctions $f(\mathcal{A}_{SH}, \eta, t)$ and $g(\mathcal{A}_{SV}, \eta, t)$, and do not include contributions from both wavefunctions. Thus, the SH and SV propagation modes are independent, and an excitation of one propagation mode does not influence wave propagation in the other mode.

2.5. Group velocity

Acoustic radiation force excitations are typically impulsive and spatially localized, and generate displacement fields described by a superposition of plane waves with a range of frequencies, phases, and propagation directions. As these waves evolve in time, the waves interfere, and the observed propagation velocity is characterized by the group velocity \vec{V} . In terms of the phase velocity v and propagation wave vector $\vec{k} = k\hat{n}$, the group velocity is given by Tsvankin (2012) as

$$\vec{V} = \frac{\partial(kv)}{\partial k_x}\hat{x} + \frac{\partial(kv)}{\partial k_y}\hat{y} + \frac{\partial(kv)}{\partial k_z}\hat{z}.$$
(33)

For plane wave propagation in the x - z plane, the components of group velocity are given in terms of the phase velocity and angle by Tsvankin (2012) as

$$V_x = v \sin \theta + \frac{\partial v}{\partial \theta} \cos \theta, \qquad V_y = 0, \qquad \text{and} \qquad V_z = v \cos \theta - \frac{\partial v}{\partial \theta} \sin \theta.$$
 (3)

Thus, the group propagation vector \widehat{N} also lies in the x - z plane and can be written as $\widehat{N} = (\sin \theta_g, 0, \cos \theta_g)^T$ where the group propagation angle θ_g is given by,

$$\theta_g = \tan^{-1} \left(\frac{V_x}{V_z} \right) = \theta + \tan^{-1} \left(\frac{1}{v} \frac{\partial v}{\partial \theta} \right).$$
(35)

For the SH propagation mode, the form of the phase velocity (28) allows the group velocity to be expressed in the form of an ellipse (Wang *et al* 2013),

$$\rho V_{SH}^2(\theta_g) = \frac{\mu_{\parallel} \,\mu_{\perp}}{\mu_{\parallel} \sin^2 \theta_g + \mu_{\perp} \cos^2 \theta_g}.$$
(36)

For the special case of an unstretched, isotropic material with $\lambda = 1$, the phase velocities v_{SH} and v_{SV} are independent of the propagation angle θ and, from (35), $\theta_g = \theta$. Then, from (34), the group velocities are equal to the phase velocities and can be expressed in terms of the shear modulus of the material using (32),

$$\mu = \rho V_{SH}^2(\lambda = 1) = \rho V_{SV}^2(\lambda = 1) = 2 \left. \frac{\partial W}{\partial I_1} \right|_{\lambda = 1} + 2 \left. \frac{\partial W}{\partial I_2} \right|_{\lambda = 1}.$$
(37)

2.6. Comparison of phase and group velocities along and across the symmetry axis

From (27) and (31), the derivative $\partial v/\partial \theta$ is zero for propagation of both the SH and SV wave modes in the directions along and across the material symmetry axis at angles $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$, respectively. Then, for these directions, the group propagation angle θ_g from (35) is equal to the phase propagation angle θ , and from (34), the group velocities are equal to the phase velocities,

$$V_{SH}(\theta_g = 0^\circ) = v_{SH}(\theta = 0^\circ), \qquad V_{SH}(\theta_g = 90^\circ) = v_{SH}(\theta = 90^\circ), V_{SV}(\theta_g = 0^\circ) = v_{SV}(\theta = 0^\circ), \qquad \text{and} \qquad V_{SV}(\theta_g = 90^\circ) = v_{SV}(\theta = 90^\circ).$$
(38)

In addition, we can compare the group velocities along and across the material symmetry axis for the SH and SV propagation modes. For propagation along the axis at $\theta = \theta_g = 0^\circ$, the phase and group velocities are equal and (27), (28), (29), (31), and (38) give

$$\rho v_{SH}^2(\theta = 0^\circ) = \rho v_{SV}^2(\theta = 0^\circ) = \rho V_{SH}^2(\theta_g = 0^\circ) = \rho V_{SV}^2(\theta_g = 0^\circ) = \mu_{\parallel}.$$
 (39)

Similarly, for propagation across the symmetry axis at $\theta = 90^{\circ}$, the phase velocities are given by

$$\rho v_{SH}^2(\theta = 90^\circ) = \mu_\perp$$
 and $\rho v_{SV}^2(\theta = 90^\circ) = \mu_\perp + \left(\frac{2}{\lambda^2} - 2\lambda\right) \frac{\partial W}{\partial I_2}.$ (40)

Then, with (38), the difference between the SH and SV group velocities is given by

$$\rho V_{SV}^2(\theta_g = 90^\circ) - \rho V_{SH}^2(\theta_g = 90^\circ) = \left(\frac{2}{\lambda^2} - 2\lambda\right) \frac{\partial W}{\partial I_2}.$$
(41)

Table 1. Material models, strain energy functions $W(I_1, I_2)$, and model parameters from Caenen *et al* (2020). The parameters have units of kPa except for λ_m which is unitless.

Material model	Strain energy $W(I_1, I_2)$	Model parameters
Arruda-Boyce	$W = C_1 \sum_{i=1}^{5} \alpha_i \lambda_m^{2-2i} \left(I_1^i - 3^i \right)$	$C_1 = 8.01$
	$\alpha_1 = \frac{1}{2}, \ \alpha_2 = \frac{1}{20}, \ \alpha_3 = \frac{11}{1050}, \ \alpha_4 = \frac{19}{7000}, \ \alpha_5 = \frac{519}{673750}$	$\lambda_m = 1.58$
Mooney-Rivlin	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)$	$C_{10} = 7.89$
		$C_{01} = -0.538$
Isihara	$W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{01}(I_2 - 3)$	$C_{10} = 5.44$
		$C_{01} = -0.523$
		$C_{20} = 0.677$

Thus, the SH and SV group velocities are different for propagation across the symmetry axis if the material is dependent on the invariant I_2 and $\lambda \neq 1$. This phenomenon is known as shear splitting and has been observed in both the experimental measurements and finite element simulations by Caenen *et al* (2020) where the tilted experimental configuration allowed excitation of both the SH and SV propagation modes.

3. Arruda-Boyce, Mooney-Rivlin, and Isihara material models

This section presents sample results for the phase and group velocities for the three material models considered by Caenen *et al* (2020), specifically, the Arruda-Boyce (1993), Mooney-Rivlin (Mooney 1940, Rivlin and Saunders 1951), and Isihara (1951) models. Each model was assumed to be incompressible. Expressions for the strain energy $W(I_1, I_2)$ are given in table 1 along with the model coefficients determined by Caenen *et al* (2020) using mechanical testing of a sample of the PVA material used in their study.

Table 2 lists expressions for the phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ calculated using (27) and (31), and the shear modulus μ calculated using (32), for each of the material models. In these expressions, the invariants I_1 and I_2 of the deformation tensor **B** are written in terms of the stretch λ using (3).

Figure 2(a) shows polar plots of the phase velocities $v_{SH}(\theta)$ (dashed black line) and $v_{SV}(\theta)$ (solid red line) calculated using the expressions in table 2 evaluated using the model coefficients listed in table 1 for the specific stretch values of $\lambda = 1.60$ and $\lambda = 2.14$. These values were chosen to illustrate the shear wave speeds corresponding to the maximum stretch of $\lambda = 2.14$ achieved in the experimental measurements of Caenen *et al* (2020), and a second stretch with $\lambda = 1.6$ roughly midway between the maximum stretch and the case for an isotropic material with $\lambda = 1$.

Figure 2(b) shows polar plots of the group velocities for the SH and SV propagation

Table 2. Phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ from (27) and (31), and shear modulus μ from (32), for the Arruda-Boyce, Mooney-Rivlin, and Isihara material models in table 1. Material model Phase velocities and shear modulus $\rho v_{SH}^2(\theta) = 2C_1 \left[\sum_{i=1}^N i\alpha_i \lambda_m^{2-2i} \left(\lambda^2 + \frac{2}{\lambda} \right)^{i-1} \right] \left(\lambda^2 \cos^2 \theta + \frac{1}{\lambda} \sin^2 \theta \right)$ Arruda-Boyce $\rho v_{SV}^2(\theta) = 2C_1 \left[\sum_{i=1}^N i\alpha_i \lambda_m^{2-2i} \left(\lambda^2 + \frac{2}{\lambda} \right)^{i-1} \right] \left(\lambda^2 \cos^2 \theta + \frac{1}{\lambda} \sin^2 \theta \right)$ $+4C_1 \left[\sum_{i=1}^N i(i-1)\alpha_i \lambda_m^{2-2i} \left(\lambda^2 + \frac{2}{\lambda}\right)^{i-2} \right] \left(\lambda^2 - \frac{1}{\lambda}\right)^2 \sin^2\theta \cos^2\theta$ $\mu = 2C_1 \left(\sum_{i=1}^N i\alpha_i \lambda_m^{2-2i} \, 3^{i-1} \right)$ $\rho v_{SH}^2(\theta) = (2\lambda^2 C_{10} + 2\lambda C_{01})\cos^2\theta + \left(\frac{2}{\lambda}C_{10} + 2\lambda C_{01}\right)\sin^2\theta$ Mooney-Rivlin $\rho v_{SV}^2(\theta) = \left(2\lambda^2 C_{10} + 2\lambda C_{01}\right)\cos^2\theta + \left(\frac{2}{\lambda} C_{10} + \frac{2}{\lambda^2} C_{01}\right)\sin^2\theta$ $\mu = 2C_{10} + 2C_{01}$ $\rho v_{SH}^2(\theta) = \left[2C_{10} + 4C_{20}\left(\lambda^2 + \frac{2}{\lambda} - 3\right)\right] \left(\lambda^2 \cos^2\theta + \frac{1}{\lambda}\sin^2\theta\right) + 2\lambda C_{01}$ Isihara $\rho v_{SV}^2(\theta) = \left[2C_{10} + 4C_{20}\left(\lambda^2 + \frac{2}{\lambda} - 3\right)\right] \left(\lambda^2 \cos^2 \theta + \frac{1}{\lambda} \sin^2 \theta\right)$ $+2C_{01}\left(\lambda\cos^2\theta+\tfrac{1}{\lambda^2}\sin^2\theta\right)+8C_{20}\left(\lambda^2-\tfrac{1}{\lambda}\right)^2\sin^2\theta\cos^2\theta$ $\mu = 2C_{10} + 2C_{01}$

modes for the same materials and stretch values as for the phase velocities in figure 2(a). As given in (36), the group velocity for the SH mode has an elliptical shape while the SV shape is more complicated and depends on the specific material model. Also, as expected from (39) and table 2, the group velocities for the SH and SV modes are the same for propagation along the material symmetry axis at $\theta_q = 0^\circ$. For propagation across the axis at $\theta_q = 90^\circ$, the group velocities are the same for the Arruda-Boyce material model because the strain energy for this model is not dependent of the invariant I_2 and $V_{SH}(\theta_q = 90^\circ) = V_{SV}(\theta_q = 90^\circ)$ from (41). However, shear splitting is observed for the Mooney-Rivlin and Isihara models because the strain energies for these materials depend on the invariant I_2 so that $V_{SH}(\theta_g = 90^\circ) \neq V_{SV}(\theta_g = 90^\circ)$. Also, from (41), $V_{SV}(\theta_g = 90^\circ) > V_{SH}(\theta_g = 90^\circ)$ because the coefficient C_{01} from table 1 is negative for both material models. These results are in agreement with the results shown in figures 5 - 9 of Caenen *et al* (2020).

As expected, the angular pattern of phase and group velocities in figures 2(a) and 2(b) becomes less elongated as the stretch decreases. In the limit of no stretch with $\lambda = 1$, the material is isotropic and does not have a symmetry axis so that there is no distinction between the SH and SV propagation modes. The phase and group velocities are equal and do not depend on the propagation direction. Polar plots of these velocities (not shown) are circular with values determined from (32) and (37) evaluated using the



Figure 2. Polar plots of the phase velocity (a) and group velocity (b) for the SH (black, dashed) and SV (red, solid) propagation modes with stretch values $\lambda = 1.60$ and $\lambda = 2.14$ for the Arruda-Boyce (left), Mooney-Rivlin (center) and Isihara (right) models evaluated using the material parameters from Caenen *et al* (2020) listed in table 1. Phase velocities are calculated using expressions (27) and (31) and are given in table 2. Group velocities are calculated from the phase velocities using (34). The bottom row (c) shows plots of the group propagation angle θ_g as a function of the phase propagation angle θ calculated using (35). In some cases, the inverse relation $\theta(\theta_g)$ and resulting group velocities are triple valued, leading to the cusp structure seen in the group velocity for the Arruda-Boyce and Isihara models.

specific expressions for the shear modulus μ given in table 2 with the material parameters from table 1. Specifically, the phase and group velocities of the unstretched material are 3.33 m/s for the Arruda-Boyce model, 3.83 m/s for the Mooney-Rivlin model, and 3.14 m/s for the Isihara model. Differences among these values occur because of the differing degree to which each material model can describe the stress-strain relation in the mechanical testing measurements, see figure 4 in Caenen *et al* (2020), and the uncertainties of the parameters determined in these measurements, see table 1 of Caenen *et al* (2020).

Figure 2(c) shows plots of the group propagation angle θ_g calculated using (35) as a function of the phase propagation angle θ . For the Arruda-Boyce and Isihara material models, the inverse relation $\theta(\theta_g)$ is triple-valued at larger values of stretch and thereby gives tripled valued group velocities as seen in the polar plots in figure 2(b).

4. Discussion

Phase and group velocities provide complementary information related to the analysis of shear wave propagation in anisotropic materials. The phase velocity provides an

analytical connection to the material model through the strain energy function, while in contrast, the group velocity is measured from experimental observations of shear wave propagation. A relation between these velocities is required to characterize the material from experimental measurements. This relation is trivial for the case of an isotropic material, but as shown in sections 2.5 and 2.6 for an anisotropic material, the phase and group velocities are equal only for wave propagation along and across the material symmetry axis. For other propagation directions, the group velocity can be derived from the angle-dependent phase velocity assuming this functional relation is known. The analytic expressions derived in this paper provide the needed relations $v_{SH}(\theta)$ and $v_{SV}(\theta)$ for the SH and SV propagation modes, and thus, allow the calculation of the group velocities $V_{SH}(\theta_g)$ and $V_{SV}(\theta_g)$ using (34) and (35).

For the SH propagation mode, the phase and group velocities can be expressed in terms of the shear moduli μ_{\parallel} and μ_{\perp} for propagation along and across the material symmetry axis using (28) and (36). These expressions are the same as for the case of a linear, elastic material with intrinsic TI structure as given by Tsvankin (2012), Carcione (2015), and Rouze *et al* (2013). In particular, the group velocity (36) has an elliptical shape and has been observed in the angular dependent measurements by Urban *et al* (2015) and Chatelin *et al* (2014) using the common experimental geometry with the excitation axis oriented perpendicular to the tracking plane as shown, for example, in figure 1(a) of Rouze *et al* (2020) or figure 2 from Chatelin *et al* (2014).

The phase velocity for the SV propagation mode is more complicated than for the SH propagation mode due to the additional terms in the angular dependence of the SV phase velocity in (31) compared to the SH phase velocity in (27). The SV group velocity is also more complicated and, in general, a closed form expression cannot be given for this velocity except for the case of propagation along and across the material symmetry axis as in (38). In particular, for propagation across the symmetry axis, the SV group velocity differs from the SH velocity and the material exhibits shear splitting with a difference in velocity that depends on the form of the strain energy W in (41). Thus, experimental observations of shear splitting allow additional, independent measurements that can help determine model parameters in the strain energy function. In addition, as shown in the plots for the Arruda-Boyce and Isihara material models in figure 2, the SV group velocity can show complicated, cusp structures where the velocity is triple valued. As seen in figure 2(c), this structure occurs when the coefficient of the $\sin^2\theta\cos^2\theta$ term in the expression for the SV phase velocity has sufficient amplitude that the group propagation angle θ_q (35) has inflection points and the inverse relation $\theta(\theta_q)$ is triple valued. Thus, observation of the cusp structure in the angular dependence of the group propagation velocity can help to determine the form of the strain energy W in the material model from the coefficients of the $\sin^2 \theta \cos^2 \theta$ terms in (31).

In addition, as indicated in (41), the presence of shear splitting will increase as the stretch λ increases. Similarly, the cusp structure that appears due to the $\sin^2 \theta \cos^2 \theta$ term in (31) will also appear only at larger values of stretch due to the λ^4 , λ^3 , and λ^2 dependence in the $\partial^2 W/\partial I_1^2$, $\partial^2 W/\partial I_2 \partial I_2$, and $\partial^2 W/\partial I_2^2$ terms, respetively. Thus,

both of these effects may not be obvious in experiments performed with smaller material deformations.

There are similarities and differences between the descriptions of shear wave propagation in the stretched, hyperelastic material considered in this study, and a linear, elastic material with an intrinsic, TI structure as considered in, for example, Tsvankin (2012), Carcione (2015), and Rouze *et al* (2013). For both types of materials, the phase velocities for the SH propagation mode can be expressed in terms of the shear moduli μ_{\parallel} and μ_{\perp} as in (28), and the corresponding group velocities are elliptical with wave speeds along and across the symmetry axis determined by μ_{\parallel} and μ_{\perp} . However, for the SV propagation mode, the expressions for the phase velocities in the two types of materials are different. For a linear, elastic, incompressible material with intrinsic TI structure, the phase velocity v_{SV} is given by Rouze *et al* (2013),

$$\rho v_{SV}^2 = \mu_{\parallel} + 4 \left(\frac{E_{\parallel}}{E_{\perp}} \mu_{\perp} - \mu_{\parallel} \right) \sin^2 \theta \cos^2 \theta \tag{42}$$

where E_{\parallel} and E_{\perp} are Young's moduli that characterize the tensile stiffness along and across the axis of the material. The differences between this expression and expression (31) for a stretched, hyperelastic material lead to an important difference for the group velocities that determine the shear splitting for propagation across the material symmetry axis. Specifically, for the intrinsic, TI material, this difference is a fixed value determined by the difference $\mu_{\parallel} - \mu_{\perp}$ of shear moduli while, in contrast, for a hyperelastic material, this difference is an independent quantity determined by the stretch λ and the term $\partial W/\partial I_2$ in (41).

A second difference for the group velocities of the SV propagation mode in the intrinsic, TI material and the stretched, hyperelastic material appears in the location of the cusp structures where the group velocities are triple valued. As shown in figure 1 of Rouze et al (2013), cusp structures in the intrinsic, TI material are centered around the angles 45° and 135° , or about the angles 0° and 90° depending on the sign of the $\sin^2\theta\cos^2\theta$ term in (42). However, for the Arruda-Boyce and Isihara materials in figure 2, the location of these structures is determined by the coefficients of the $\sin^2\theta$ and $\sin^2\theta\cos^2\theta$ terms in (31). One result of this difference is that Knight *et al* (2021) observed SV wave propagation at angles near the material symmetry axis in their in vivo measurements in muscle with a tilted excitation configuration. However, Caenen et al (2020) did not observe clear SV wave propagation in this location in their measurements with a 45° tilt angle. This difference can be explained by the location of the cusp structure in the results for the group velocities in the Arruda-Boyce and Isihara materials shown in figure 2 where the structure is observed at angles near 15° , and thus, was not included in the angular range interrogated in the experiment by Caenen *et al* (2020).

Finally, it is also important to consider the relative amplitude of the shear wave signals for both the SH and SV modes as a function of propagation direction. The primary factor in the determination of the wave amplitude is the degree to which each mode is excited. As shown, for example, in figure 1 of Rouze *et al* (2020), for the

common experimental geometry in which the excitation axis is oriented perpendicular to the observation plane, only the SH mode is excited, and SV waves will not be observed. Similarly, for the measurements reported by Knight *et al* (2021) in muscle, a small tilt angle of 11° was realized, and the amplitude of the SV mode was not sufficiently large to be observed for propagation in a direction across the material symmetry axis.

5. Conclusion

This paper derives analytic expressions for angular-dependent phase velocities $v_{SH}(\theta)$ and $v_{SV}(\theta)$ of the SH and SV shear wave propagation modes in an incompressible, hyperelastic material with finite, uniaxial stretch. The phase velocities are determined by constructing the equations of motion for plane wave propagation using the Cauchy stress tensor derived from the strain energy of the material. Group propagation velocities $V_{SH}(\theta_q)$ and $V_{SV}(\theta_q)$ such as those measured in experimental observations of shear wave propagation are calculated as a function of propagation direction θ_g from the phase velocities. Results are presented for the Arruda-Boyce, Mooney-Rivlin, and Isihara material modes using model coefficients determined by Caenen et al (2020). Results for the Mooney-Rivlin and Isihara material models show shear splitting with $V_{SV} > V_{SH}$ at larger values of stretch for propagation across the material symmetry axis. Results for the Arruda-Boyce and Isihara models show cusp structures with triple-valued group velocities at larger values of material stretch for specific propagation directions. For experimental configurations in which both SH and SV wave motions are observed, these features provide additional data that can be used to improve the use of shear wave analysis techniques for the characterization of elastic properties of materials.

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