A characterization of the Coxeter cap

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Abstract

In this paper, we complete the classification of the caps in $\mathrm{PG}(n,q)$ having the property that on every tangent line L, there exists a unique point distinct from the tangency point though which there is at least one secant line. The examples include the Coxeter cap in $\mathrm{PG}(5,3)$ related to the Mathieu group M_{12}, a set of three noncollinear points in $\mathrm{PG}(2,q)$ and some examples related to hyperovals of projective planes.

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1 Introduction

In 1958, Coxeter [3] studied a set $\mathcal{K}$ of 12 points of the projective space $\mathrm{PG}(5,3)$ that satisfies some marvellous properties, among which we have:

(a) The stabilizer of $\mathcal{K}$ inside the group $PGL(6,3)$ is isomorphic to the Mathieu group M_{12}.

(b) Every five distinct points of $\mathcal{K}$ generate a hyperplane intersecting $\mathcal{K}$ in exactly six points.

(c) The point-line geometry defined on $\mathcal{K}$ by the hyperplane intersections of size 6 is a Steiner system of type $S(5,6,12)$ (necessarily) isomorphic to the Witt design on 12 points.

(d) For every tangent line L, there exists a unique point on $L \setminus \mathcal{K}$ that is on a secant line. Moreover, the secant line through that point is unique.

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We note that property (b) implies that $\mathcal{K}$ is a so-called *cap*, that is, a set of points of a projective space intersecting each line in at most two points. Having a cap X in a projective space, we then define *external*, *tangent* or *secant lines* as lines meeting X in respectively 0, 1 or 2 points. If L is a tangent line, then the unique point in $L \cap X$ is called the *tangency point*. Any set of points in $\mathrm{PG}(5,3)$ projectively equivalent to the above-mentioned set of 12 points is called a *Coxeter cap*.

Among the above-mentioned properties (a), (b), (c) and (d), the last one is the one which is of interest in the present paper. We wish to classify all caps in finite projective spaces that satisfy (d), with omission of the requirement that the secant line through the point of $L \setminus \mathcal{K}$ needs to be unique. We also succeed in this goal. The following is our main result.

Theorem 1.1. *Let $\mathcal{K}$ be a nonempty cap in the projective space $\mathrm{PG}(n,q)$, $q \geq 3$, satisfying the following property.*

- ($*$) *There is a tangent line and on every tangent line, there is a unique point different from the tangency point through which there is at least one secant line.*

Then one of the following cases occurs.

1. *$n = 2$ and $\mathcal{K}$ is a set of three noncollinear points;*
2. *$n = 3$, q is even and $\mathcal{K}$ consists of a hyperoval in a plane of $\mathrm{PG}(3,q)$, plus one extra point not in that plane;*
3. *$n = 4$, q is even and $\mathcal{K}$ is the union of two hyperovals whose carrying planes meet in a point belonging to both hyperovals.*
4. *$n = 5$, $q = 3$ and $\mathcal{K}$ is a Coxeter cap in $\mathrm{PG}(5,3)$.*

The condition on the existence of tangent lines in Theorem 1.1 is in some sense not essential. If we would omit that requirement, then the extra examples that arise are nonempty sets intersecting each line in either 0 or 2 points. All such sets are well known in finite geometry (and we have therefore already excluded this case from Theorem 1.1). They are either the singleton in $\mathrm{PG}(0,q)$, pairs of points in $\mathrm{PG}(1,q)$, hyperovals in $\mathrm{PG}(2,q)$ with q even, or the complements of hyperplanes in $\mathrm{PG}(n,q)$ with $n \geq 3$ and $q = 2$.

What happens if we omit the condition $q \geq 3$ in Theorem 1.1? As we will see in Section 3, all examples with $q = 2$ can be described in a uniform way. They are the sets X of points of $\mathrm{PG}(n,2)$ that satisfy the following properties:

(a) X is properly contained in the complement of a hyperplane;

(b) for every point x of $\mathrm{PG}(n,2)$, there exists a subset $Y \subseteq X$ with $|Y| \in \{1,2,3\}$ such that $x \in \langle Y \rangle$.

There are plenty of point sets satisfying (a) and (b), e.g. by starting from the complement of a hyperplane of $\mathrm{PG}(n, 2)$ and removing any positive number of points such that (b) is still satisfied. In view of this, it seems for $q = 2$ not possible to give a classification for general values of n beyond the description that we have given in (a) and (b) above. We have therefore also opted not to include the case $q = 2$ in Theorem 1.1. It also seems that the classification for the case $q = 2$ is most naturally obtained in a more general setting, and this is also the approach we will follow in Section 3.

An incomplete classification of the caps in $\mathrm{PG}(n, q)$ satisfying property $(*)$ of Theorem 1.1 was already obtained in [7]. These sets of points were studied in [7] because of their connection with (linear representations of) near hexagons. The reason why we are now able to complete the classification, more than 20 years after [7], is because of some recent breakthrough results in the theory of near polygons, more precisely, the derivation of a new divisibility condition [5] and a new inequality [6] that must be satisfied by the parameters of near hexagons with an order. This inequality and divisibility condition have been derived in [5, 6] by means of algebraic combinatorial arguments using certain matrices associated with the geometry.

These new results allow to simplify several of the early results in the theory of near polygons, to extend classification results for near hexagons to a more general setting, and to show the nonexistence of a number of near hexagons whose existence had been open for many years. In the present paper, we offer another application of these results to certain caps in projective spaces, hereby completing a classification that was initiated 20 years ago in [7].

In Section 2.2, we describe the connection between near polygons and the caps of $\mathrm{PG}(n, q)$ that satisfy property $(*)$ of Theorem 1.1. The recent results in the theory of near polygons that are essential for our treatment are the Propositions 2.1 and 2.2 of Section 2.1. The completion of the classification will take place in Section 5. In order to achieve this goal, we need to recall several properties of the caps that were already obtained in [7]. These will be recalled in Sections 4, 5.2 and 5.3.

2 Preliminaries

2.1 Near polygons

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *partial linear space* if every two distinct points are incident with at most one line. A point-line geometry is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points, and if every point is incident with precisely $t + 1$ lines. A point-line geometry is called *thin* if every line is incident with exactly two points. Distances in a point-line geometry $\mathcal{S}$ are always measured in its collinearity graph. If O_1 and O_2 are two objects of $\mathcal{S}$, being points or nonempty sets of points, then $\mathrm{d}(O_1, O_2)$ denotes the distance between O_1 and O_2. If O is an object and $i \in \mathbb{N}$, then $\Gamma_i(O)$ denotes the set of points at distance i from O.

A partial linear space is called a *generalized quadrangle* (GQ) if there exist two lines that have no points in common, and if for every non-incident point-line pair (x, L), there exists a unique point on L collinear with x. A partial linear space is called a *degenerate generalized quadrangle* if there are at least two lines, any two lines have a point in common and for every non-incident point-line pair (x, L), there exists a unique point on L collinear with x. Degenerate generalized quadrangles are *line pencils* for which there exists a specific point that is collinear with all other points and incident with all lines. A finite generalized quadrangle $\mathcal{Q}$ of order (s, t) contains $(s + 1)(st + 1)$ points and $(t + 1)(st + 1)$ lines. The diameter of $\mathcal{Q}$ is moreover equal to 2, and $|\Gamma_0(x)| = 1$, $|\Gamma_1(x)| = s(t + 1)$, $|\Gamma_2(x)| = s^2t$ for any point x of $\mathcal{Q}$. For more background information on generalized quadrangles, we refer to [14].

A partial linear space $\mathcal{S}$ is called a *near polygon* if $\mathcal{S}$ has finite diameter and if for every point-line pair (x, L), there exists a unique point $\pi_L(x)$ on L that is nearest to x. If $d \in \mathbb{N}$ is the diameter of $\mathcal{S}$, then $\mathcal{S}$ is called a *near* $2d$*-gon*. A near 0-gon has a unique point and no lines, a near 1-gon is a line and the near quadrangles are the possibly degenerate generalized quadrangles. The thin near polygons are precisely the bipartite graphs of finite diameter. As a near polygon $\mathcal{S}$ is a partial linear space, we can and often will identify a line L of $\mathcal{S}$ with the set of points incident with L. Near polygons were introduced by Shult and Yanushka in [15].

Let $\mathcal{S}$ be a near polygon. A set X of points of $\mathcal{S}$ is called a *subspace* if every line that has two of its points in X has all its points in X. For every nonempty subspace X, a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ can be defined whose points are the elements of X and whose lines are the lines of $\mathcal{S}$ that have all their points in X. A set X of points of $\mathcal{S}$ is called *convex* if for every two points x and y of X, all points on a shortest path between x and y are also contained in X. If X is a nonempty convex subspace of $\mathcal{S}$, then $\mathcal{S}_X$ obviously is also a near polygon. If X is a nonempty convex subspace of $\mathcal{S}$ for which $\mathcal{S}_X$ is a (nondegenerate) generalized quadrangle, then X is called a *quad*.

The following proposition was recently proved in [6, Theorem 1.3].

Proposition 2.1. *Suppose $\mathcal{S}$ is a finite near hexagon of order (s, t), $s \geq 2$. Let x and y be two points of $\mathcal{S}$ at distance 3 from each other. Let N denote the number of shortest paths connecting x and y, and put $\overline{t_2} := \frac{N}{t+1} - 1$. Then*

$$\overline{t_2}(s^2 + s + 1) - s^3 \leq t \leq s^3 + \overline{t_2}(s^2 - s + 1).$$

In case every two points at distance 2 have exactly $t_2 + 1$ common neighbours, the near hexagon of order (s, t) is said to be *regular with parameters* (s, t, t_2) and the upper bound in Proposition 2.1 reduces to $t \leq s^3 + t_2(s^2 - s + 1)$. This bound is known as the (Haemers-)Mathon bound, see [2, 9, 11, 12]. In case every two points at distance 2 have a unique common neighbour, the near hexagon of order (s, t) is said to be a *generalized hexagon* of order (s, t) and the upper bound in Proposition 2.1 reduces to $t \leq s^3$. This bound is known as the Haemers-Roos inequality, see [10].

We also mention here another restriction for near hexagons with an order recently proved in Theorem 1(2) of [5].

Proposition 2.2. *Suppose $\mathcal{S}$ is a finite near hexagon of order (s,t) having v points. Then the number*

$$\frac{s^5 v}{(s+1)^2(s-1)(s^2+1)+st(s-1)(s+1)^2+v}$$

is integral.

The following proposition, taken from [6, Proposition 2.3], can readily be derived from [4, Theorem 1.2].

Proposition 2.3. *Let $\mathcal{S}$ be a finite near hexagon of order (s,t) having v points. Then $|\Gamma_2(x)| = \frac{v}{s+1} - 1 + st(s-1)$ for every point x of $\mathcal{S}$.*

The following proposition is precisely Proposition 2.6 of [15].

Proposition 2.4. *Suppose $\mathcal{S}$ is a near polygon having the property that every line is incident with at least three points. Then one of the following cases occurs for a point-quad pair (x,Q) of $\mathcal{S}$:*

(1) *There exists a unique point $\pi_Q(x) \in Q$ nearest to x. In this case, $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_Q(x)) + \mathrm{d}(\pi_Q(x),y)$ for every point $y \in Q$.*

(2) *The points in Q nearest to x form an ovoid O_x of $\mathcal{S}_Q$.*

By an *ovoid* of a generalized quadrangle $\mathcal{Q}$, we mean a set of points having a unique point in common with each line. With a *fan of ovoids* of $\mathcal{Q}$, we mean a collection of ovoids of $\mathcal{Q}$ partitioning its point set. With a *rosette of ovoids* of $\mathcal{Q}$, we mean a collection of ovoids of $\mathcal{Q}$ through a distinguished point x that partitions the set of points of $\mathcal{Q}$ at distance 2 from x. The point x is then called the *center* of the rosette. If $\mathcal{Q}$ is a finite generalized quadrangle of order (s,t), then every ovoid of $\mathcal{Q}$ contains $1+st$ points, every fan of ovoids contains $s+1$ elements and every rosette of ovoids contains s elements.

If case (1) occurs in Proposition 2.4, then the point x is called *classical* with respect to Q. If case (2) occurs, then the point x is called *ovoidal* with respect to Q. Points at distance at most 1 from Q are always classical with respect to Q. If $\mathcal{S}$ is a near $2d$-gon, then the maximal distance from a point of $\mathcal{S}$ to Q is at most $d-1$ and points at distance $d-1$ from Q are necessarily ovoidal with respect to Q. So, we have the following.

Proposition 2.5. *Suppose $\mathcal{S}$ is a near hexagon having the property that every line is incident with at least three points. If (x,Q) is a point-quad pair, then x is classical with respect to Q if $\mathrm{d}(x,Q) \le 1$ and ovoidal with respect to Q if $\mathrm{d}(x,Q) = 2$.*

The following two propositions were proved in [2, Section(b)].

Proposition 2.6. *Suppose $\mathcal{S}$ is a near polygon having at least three points on each line, and let Q be a quad of $\mathcal{S}$. Then the following hold:*

(1) *If a line L contains points at distance i and $i+1$ from Q for some $i \in \mathbb{N}$, then L has a unique point at distance i from Q.*

(2) *If a line L contains two distinct points x and y at distance i from Q where $i \in \mathbb{N}$, then x and y are both classical or both ovoidal with respect to Q.*

Proposition 2.7. *Suppose $\mathcal{S}$ is a near polygon having at least three points on each line. Let Q be a quad of $\mathcal{S}$ and L a line at distance i from Q. Then precisely one of the following cases occurs.*

(1) *All points of L lie at distance i from Q and are classical with respect to Q. Then the points $\pi_Q(x)$, $x \in L$, are mutually distinct and form a line of $\mathcal{S}_Q$.*

(2) *All points of L lie at distance i from Q and are ovoidal with respect to Q. Then the ovoids O_x, $x \in L$, form a fan of ovoids of $\mathcal{S}_Q$.*

(3) *L contains a unique point at distance i from Q and all points of L are classical with respect to Q. Then all points $\pi_Q(x)$, $x \in L$, are equal.*

(4) *L contains a unique point at distance i from Q and all points of L are ovoidal with respect to Q. Then all ovoids O_x, $x \in L$, coincide.*

(5) *L contains a unique point y at distance i from Q, y is classical with respect to Q and every point of $L \setminus \{y\}$ is ovoidal with respect to Q. Then the ovoids O_x, $x \in L \setminus \{y\}$, form a rosette of ovoids of $\mathcal{S}_Q$ with center $\pi_Q(y)$.*

2.2 Linear representations of near polygons

Let $\mathrm{PG}(n,q)$ be a projective space of dimension $n \in \mathbb{N}$ embedded as a hyperplane in a projective space $\mathrm{PG}(n+1,q)$. For every set $\mathcal{K}$ of points of $\mathrm{PG}(n,q)$, we can define the following point-line geometry $T_n^*(\mathcal{K})$:

- the points of $T_n^*(\mathcal{K})$ are the points of $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$;
- the lines of $T_n^*(\mathcal{K})$ are those lines of $\mathrm{PG}(n+1,q)$ not contained in $\mathrm{PG}(n,q)$ through some point of $\mathcal{K}$;
- incidence is containment.

Then $T_n^*(\mathcal{K})$ is a partial linear space of order $(q-1, |\mathcal{K}|-1)$.

With every point p of $\mathrm{PG}(n,q)$, we associate in the following way an element $i_{\mathcal{K}}(p) \in \mathbb{N} \cup \{+\infty\}$ which is called the *(generating) index* of p (*with respect to* $\mathcal{K}$), or also the *$\mathcal{K}$-index*:

- If $p \notin \langle \mathcal{K} \rangle$, then $i_{\mathcal{K}}(p) = +\infty$.
- If $p \in \langle \mathcal{K} \rangle$, then $i_{\mathcal{K}}(p)$ is the smallest number of points of $\mathcal{K}$ that generate a subspace containing p.

The following propositions and corollary are taken from Section 4 of [7].

Proposition 2.8. *If x and y are two different points of $T_n^*(\mathcal{K})$ and z is the intersection point of the line xy with* $\mathrm{PG}(n,q)$, *then* $\mathrm{d}(x,y) = i_{\mathcal{K}}(z)$, *where* $\mathrm{d}(\cdot,\cdot)$ *denotes the distance function in (the collinearity graph of) $T_n^*(\mathcal{K})$.*

Proposition 2.9. *$T_n^*(\mathcal{K})$ is a near polygon if and only if $\mathcal{K} \neq \emptyset$ and for every line L of* $\mathrm{PG}(n,q)$ *containing a point $u \in \mathcal{K}$, there exists a unique point in $L \setminus \{u\}$ with smallest $\mathcal{K}$-index.*

As a consequence of Propositions 2.8 and 2.9, we have

Corollary 2.10. *If $T_n^*(\mathcal{K})$ is a near polygon, then $\mathcal{K}$ is a cap and $\langle \mathcal{K} \rangle = \mathrm{PG}(n,q)$.*

Proposition 2.11. *$T_n^*(\mathcal{K})$ is a near hexagon if and only if the following conditions are satisfied:*

(1) *$\mathcal{K}$ is a cap of* $\mathrm{PG}(n,q)$.

(2) *There is a tangent line in* $\mathrm{PG}(n,q)$ *and on every such tangent line, there is a unique point different from the tangency point through which there is at least one secant line.*

2.3 Some examples of linear representations

Let $\mathbb{F}_3^{12}$ denote the 12-dimensional vector space over the finite field $\mathbb{F}_3 = \{0,1,-1\}$ of order 3 whose vectors are row matrices of length 12 with entries in $\mathbb{F}_3$. The six rows of the matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

generate a 6-dimensional subspace C of $\mathbb{F}_3^{12}$ which is called the *extended ternary Golay code*. By deleting one coordinate position, one gets a code (a subspace of $\mathbb{F}_3^{11}$) which was discovered by Golay [8].

Let $\bar{e}_i$ with $i \in \{1,2,\ldots,12\}$ denote the row vector all of whose entries are 0 except for the ith one which is equal to 1. Let $\mathbb{E}_1$ be the partial linear space whose points are all the cosets of C and whose lines are all the triples of the form $\{\bar{v}+C, \bar{v}+\bar{e}_i+C, \bar{v}-\bar{e}_i+C\}$ with $\bar{v} \in \mathbb{F}_3^{12}$ and $i \in \{1,2,\ldots,12\}$, with incidence being containment. The following proposition is due to Shult and Yanushka, see [15, pp. 30–33].

Proposition 2.12. *$\mathbb{E}_1$ is a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$.*

After fixing some reference system, the 12 columns of the matrix M define a set $\mathcal{K}^*$ of 12 points of the projective space $\mathrm{PG}(5,3)$. We regard this projective space as a hyperplane of the projective space $\mathrm{PG}(6,3)$. Proposition 2.13 below was proved in Section 6.5 of [4].

Proposition 2.13. (1) *The maximal $\mathcal{K}^*$-index of a point of* $\mathrm{PG}(5,3)$ *is equal to* 3.

(2) *If L is a line of* $\mathrm{PG}(5,3)$ *through a point x of $\mathcal{K}^*$, then $L \setminus \{x\}$ contains a unique point with smallest $\mathcal{K}^*$-index.*

(3) *$T_5^*(\mathcal{K}^*)$ is a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$.*

The geometries $\mathbb{E}_1$ and $T_n^*(\mathcal{K}^*)$ are thus two regular near hexagons with parameters $(s,t,t_2) = (2,11,1)$. It can be shown that both geometries are isomorphic. In fact, by Brouwer [1], we know that there exists up to isomorphism a unique regular near hexagon with these parameters.

A *hyperoval* of the projective plane $\mathrm{PG}(2,q)$ is a set of $q+2$ points meeting each line in either 0 or 2 points. Hyperovals can only exist if q is even. If $\mathcal{C}$ is an irreducible conic in a Desarguesian projective plane $\mathrm{PG}(2,q)$ with $q = 2^h$ even, then all tangent lines to $\mathcal{C}$ meet in a common point, which is called the *nucleus* of $\mathcal{C}$. The conic $\mathcal{C}$ together with its nucleus is then an example of a hyperoval. The following proposition taken from [14, 3.1.3] follows from Propositions 2.8 and 2.9. Indeed, these propositions imply that $T_2^*(\mathcal{O})$ is a near polygon with diameter 2 that cannot be a degenerate generalized quadrangle as the geometry has an order.

Proposition 2.14. *Suppose $\mathcal{O}$ is a hyperoval of a projective plane* $\mathrm{PG}(2,q)$*, q even, which is embedded as a hyperplane in the projective space* $\mathrm{PG}(3,q)$*. Then $T_2^*(\mathcal{O})$ is a generalized quadrangle of order $(q-1,q+1)$.*

The projective plane $\mathrm{PG}(2,4)$ has up to isomorphism a unique hyperoval, namely an irreducible conic union its nucleus. The following result was proved by Payne [13, VI.1].

Proposition 2.15. *Let $\mathcal{O}$ be a hyperoval of the projective plane* $\mathrm{PG}(2,4)$ *which is embedded as a hyperplane in the projective space* $\mathrm{PG}(3,4)$*. Then every ovoid of the generalized quadrangle $T_2^*(\mathcal{O})$ is of the form $\pi \setminus \mathrm{PG}(2,4)$ where π is a plane of* $\mathrm{PG}(3,4)$ *intersecting* $\mathrm{PG}(2,4)$ *in a line disjoint from $\mathcal{O}$. As a consequence, any two distinct ovoids of $T_2^*(\mathcal{O})$ meet in either* 0 *or* 4 *points, and $T_2^*(\mathcal{O})$ has no rosettes of ovoids.*

3 The case $q = 2$

In this section, we classify nonempty sets $\mathcal{K}$ of points of $\mathrm{PG}(n,2)$ that satisfy the following property:

($*$) For every line L of $\mathrm{PG}(n,2)$ containing a point u of $\mathcal{K}$, there exists a unique point of $L \setminus \{u\}$ with smallest $\mathcal{K}$-index.

By Proposition 2.9, this is equivalent with demanding that $T_n^*(\mathcal{K})$ is a near polygon. We hereby suppose that $T_n^*(\mathcal{K})$ lives in a projective space $\mathrm{PG}(n+1,2)$ that contains $\mathrm{PG}(n,2)$ as a hyperplane.

Proposition 3.1. *Let α be a hyperplane of* $\mathrm{PG}(n,2)$ *and* $\mathcal{K}$ *a set of points of* $\mathrm{PG}(n,2) \setminus \alpha$ *generating* $\mathrm{PG}(n,2)$. *Then* $T_n^*(\mathcal{K})$ *is a thin near polygon, or equivalently* $\mathcal{K}$ *satisfies property* $(*)$.

Proof. The fact that $\langle \mathcal{K} \rangle = \mathrm{PG}(n,2)$ implies by Proposition 2.8 that $T_n^*(\mathcal{K})$ has finite diameter. So, it suffices to prove that the collinearity graph of $T_n^*(\mathcal{K})$ is bipartite. In $\mathrm{PG}(n+1,2)$, there are three hyperplanes through α, namely $\mathrm{PG}(n,2)$ and two others, say Π_1 and Π_2. Every line of $\mathrm{PG}(n+1,2)$ containing a point of $\mathcal{K}$ and not contained in $\mathrm{PG}(n,2)$ contains a unique point of Π_1 and a unique point of Π_2. So, the collinearity graph of $T_n^*(\mathcal{K})$ is bipartite with bipartite parts $\Pi_1 \setminus \alpha$ and $\Pi_2 \setminus \alpha$. □

We now show that also the converse of Proposition 3.1 is true.

Proposition 3.2. *Let* $\mathcal{K}$ *be a nonempty set of points of* $\mathrm{PG}(n,2)$ *satisfying property* $(*)$ *(or equivalently, such that* $T_n^*(\mathcal{K})$ *is a near polygon). Then* $\langle \mathcal{K} \rangle = \mathrm{PG}(n,2)$ *and* $\mathcal{K} \subseteq \mathrm{PG}(n,2) \setminus \alpha$ *for some hyperplane* α *of* $\mathrm{PG}(n,2)$.

Proof. By Corollary 2.10, $\langle \mathcal{K} \rangle = \mathrm{PG}(n,2)$ and so there exists a set $U = \{p_1, p_2, \ldots, p_{n+1}\}$ of $n+1$ points of $\mathcal{K}$ generating $\mathrm{PG}(n,2)$. We can choose a reference system in $\mathrm{PG}(n,2)$ in such a way that the point p_i with $i \in \{1,2,\ldots,n+1\}$ coincides with the point $(0,\ldots,0,1,0,\ldots,0)$ all whose coordinates are equal to 0, except for the ith one which is equal to 1. Partition the point set of $\mathrm{PG}(n,2)$ in two parts X_0 and X_1 such that X_0 consists of all points with even weight and X_1 consists of all points with odd weight (with respect to the chosen reference system). Then X_0 is a hyperplane of $\mathrm{PG}(n,2)$. We also partition the point set of $\mathrm{PG}(n,2)$ in two sets X_0' and X_1' such that X_0' consists of all points with even $\mathcal{K}$-index and X_1' consists of all points with odd $\mathcal{K}$-index.

We prove that if p is a point of $\mathrm{PG}(n,2)$, then either $p \in X_0 \cap X_0'$ or $p \in X_1 \cap X_1'$. We prove this by induction on the weight w of p. If $w = 1$, then $p \in U$ also has index 1 as $U \subseteq \mathcal{K}$, implying that $p \in X_1 \cap X_1'$. If $w = 2$, then p has index 2 since $\mathcal{K}$ is a cap (by Corollary 2.10) and $U \subseteq \mathcal{K}$. Suppose therefore that $w \geq 3$, and let u be a point of U such that the unique third point x on the line up has weight $w-1$. Take $\epsilon \in \{0,1\}$ having the same parity as w. Then $p \in X_\epsilon$ and $x \in X_{1-\epsilon}$. By the induction hypothesis, we know that $x \in X_{1-\epsilon} \cap X_{1-\epsilon}'$. By applying Property $(*)$ to the line ux, we also see that p and x belong to distinct sets in the collection X_0', X_1'. So, p belongs to X_ϵ' and we have $p \in X_\epsilon \cap X_\epsilon'$ as we needed to prove.

As $p \in X_0 \cap X_0'$ or $p \in X_1 \cap X_1'$ for every point p of $\mathrm{PG}(n,2)$, we have $X_0 = X_0'$ and $X_1 = X_1'$. As all points of $\mathcal{K}$ have index 1, they have odd weight and so belong to X_1. So, $\mathcal{K}$ is contained in $\mathrm{PG}(n,2) \setminus \alpha$, where α is the hyperplane X_0 of $\mathrm{PG}(n,2)$. □

4 Basic properties of the caps

Our goal here is thus to classify all sets $\mathcal{K}$ in a finite Desarguesian projective space $\mathrm{PG}(n,q)$ with $n \geq 2$ for which the following two properties are satisfied:

($NH1$) $\mathcal{K}$ is a cap;

($NH2$) there is at least one tangent line, and on every tangent line, there is a unique point different from the tangency point through which there is at least one secant line.

By Proposition 2.11, this is equivalent with demanding that $T_n^*(\mathcal{K})$ is a near hexagon. We hereby suppose that $T_n^*(\mathcal{K})$ lives in a projective space $\mathrm{PG}(n+1,q)$ that contains $\mathrm{PG}(n,q)$ as a hyperplane.

By Section 3 (see also the discussion in Section 1), we precisely know how the sets $\mathcal{K}$ look like in the case that $q=2$. In fact, there are then an infinite number of examples and no bound on the dimension n of $\mathrm{PG}(n,q)$. In the sequel, we will therefore assume that $q \geq 3$. The following lemmas are taken from [7].

Lemma 4.1. *Every plane of* $\mathrm{PG}(n,q)$ *intersects* $\mathcal{K}$ *in either* 0, 1, 2, 3 *or* $q+2$ *points. If there exists a plane* α *such that* $|\alpha \cap \mathcal{K}| = q+2$, *then* q *is even and* $\alpha \cap \mathcal{K}$ *is a hyperoval of* α.

A plane of $\mathrm{PG}(n,q)$ is called *thick* whenever it intersects $\mathcal{K}$ in exactly $q+2$ points.

Lemma 4.2. (1) *If* β *is a* 3*-dimensional subspace of* $\mathrm{PG}(n,q)$ *through a thick plane, then* $|\beta \cap \mathcal{K}| \in \{q+2, q+3\}$.

(2) *Two thick planes cannot meet in a line.*

(3) *Two thick planes cannot meet in a point not belonging to* $\mathcal{K}$.

(4) *If two thick planes* α_1 *and* α_2 *intersect in a point of* $\mathcal{K}$, *then the points of* $\mathcal{K}$ *in the* 4*-dimensional subspace through* α_1 *and* α_2 *are all contained in* $\alpha_1 \cup \alpha_2$.

(4) *If two thick planes* α_1 *and* α_2 *are disjoint, then the points of* $\mathcal{K}$ *in the* 5*-dimensional subspace through* α_1 *and* α_2 *are all contained in* $\alpha_1 \cup \alpha_2$.

If Q is a quad of $T_n^*(\mathcal{K})$, then we denote by $\mathcal{S}_Q$ the subgeometry of $T_n^*(\mathcal{K})$ defined on Q by those lines of $T_n^*(\mathcal{K})$ that have all their points in Q. Recall that $\mathcal{S}_Q$ is a generalized quadrangle.

Lemma 4.3. (1) *Let* L *be a secant line of* $\mathrm{PG}(n,q)$ *that is not contained in a thick plane. If* α *is a plane of* $\mathrm{PG}(n+1,q)$ *intersecting* $\mathrm{PG}(n,q)$ *in* L, *then* $Q := \alpha \setminus L$ *is a quad of* $T_n^*(\mathcal{K})$ *for which* $\mathcal{S}_Q$ *is a* $(q \times q)$*-grid.*

(2) *Let* α *be a thick plane of* $\mathrm{PG}(n,q)$ *and* β *a* 3*-dimensional subspace through* α *not contained in* $\mathrm{PG}(n,q)$. *Then* $Q := \beta \setminus \alpha$ *is a quad of* $T_n^*(\mathcal{K})$ *for which* $\mathcal{S}_Q \cong T_2^*(\alpha \cap \mathcal{K})$.

Lemma 4.4. *Two points* x *and* y *of* $T_n^*(\mathcal{K})$ *at mutual distance* 2 *have either* 2 *or* $q+2$ *common neighbours and are contained in a unique quad which is either a* $(q \times q)$*-grid or isomorphic to* $T_2^*(\mathcal{O})$ *for some hyperoval* $\mathcal{O}$ *of* $\mathrm{PG}(2,q)$. *The latter case can only occur if* q *is even. All quads of* $T_n^*(\mathcal{K})$ *are as obtained in Lemma 4.3. If there are no thick planes, then all quads are grids and the near hexagon* $T_n^*(\mathcal{K})$ *is regular.*

Lemma 4.5. *If $k = |\mathcal{K}|$ and N is the total number of thick planes of* $\mathrm{PG}(n,q)$*, then*

$$\frac{1}{2}Nq(q^2-1) + \frac{q^n-1}{q-1} - (k-1) = \frac{1}{2}(q-1)(k-1)(k-2).$$

Proof. This was proved in Lemma 6.6 of [7] by a combinatorial reasoning in $\mathrm{PG}(n,q)$, not invoking the associated near hexagon $T_n^*(\mathcal{K})$. We now show that the equality is also equivalent with the equality of Proposition 2.3 applied to the near hexagon $T_n^*(\mathcal{K})$. We know that $T_n^*(\mathcal{K})$ is a near hexagon of order $(s,t) = (q-1, k-1)$ having $v = q^{n+1}$ points. By Proposition 2.3, we then know that

$$|\Gamma_2(x)| = q^n - 1 + (q-1)(q-2)(k-1)$$

for every point x of $T_n^*(\mathcal{K})$. By Proposition 2.8, we also know that

$$|\Gamma_2(x)| = (q-1)M,$$

where M is the total number of points of $\mathrm{PG}(n,q)$ that have index 2 with respect to $\mathcal{K}$. Hence,

$$M = \frac{q^n-1}{q-1} + (q-2)(k-1).$$

The number of pairs $(\{x,y\},z)$, where x and y are two distinct points of $\mathcal{K}$ and $z \in xy \setminus \{x,y\}$ is equal to $\binom{k}{2}(q-1) = \frac{1}{2}k(k-1)(q-1)$. We now count this number in another way. If $(\{x,y\},z)$ is such a pair, then z necessarily has index 2. By Lemmas 4.1 and 4.2(1), we then know that there are either 1 or $\frac{q+2}{2}$ secant lines through z, and that in the latter case there exists a unique thick plane containing these $\frac{q+2}{2}$ secant lines. The number of points with index 2 that are contained in $\frac{q+2}{2}$ secant lines is therefore equal to $N((q^2+q+1)-(q+2)) = N(q^2-1)$ and the number of points with index 2 that are contained in a unique secant line is equal to $M - N(q^2-1)$. So, the number of suitable pairs is also equal to

$$N(q^2-1)\cdot\frac{q+2}{2} + (M - N(q^2-1))\cdot 1 = M + N\frac{(q^2-1)q}{2}.$$

So, we have

$$\frac{1}{2}k(k-1)(q-1) = M + N\frac{(q^2-1)q}{2} = \frac{q^n-1}{q-1} + (q-2)(k-1) + N\frac{(q^2-1)q}{2}.$$

The latter equality is equivalent to the one mentioned in the lemma. $\square$

The following can be derived from Lemma 4.5, see [7, Corollary 6.7].

Lemma 4.6. *The following congruences regarding the number $k = |\mathcal{K}|$ hold:*

(1) $k \equiv n+1 \pmod{q-1}$;

(2) $k \equiv 2 \pmod q$ *or* $k \equiv 3 \pmod q$*;*

(3) $(q+1) \mid (k-1)(k-3)$ *if* n *is even and* $(q+1) \mid (k-2)^2$ *if* n *is odd.*

Remark. In [7], it was claimed that k is congruent to either 1 or 3 modulo $q+1$ if n is even, but this seems to be wrongly derived from the fact that $(q+1) \mid (k-1)(k-3)$. E.g., $q = 32$ and $k = 12$ would form a counter example.

5 Classification of the caps

5.1 Preliminaries

As explained in the beginning of Section 4, we may suppose that $q \geq 3$.

Lemma 5.1. *If there are thick planes, then* $n \geq 3$*,* $q \geq 4$ *is even and* $k = |\mathcal{K}| \geq q + n$*.*

Proof. Let α be a thick plane. As $\alpha \cap \mathcal{K}$ is a hyperoval, we have $q \geq 4$ is even.

If $n = 2$, then $T_n^*(\mathcal{K})$ is a generalized quadrangle by Proposition 2.14, which is not the case here. So, $n \geq 3$.

From Corollary 2.10, we know that $\langle \mathcal{K} \rangle = \mathrm{PG}(n, q)$. As $|\alpha \cap \mathcal{K}| = q + 2$, this implies that $|\mathcal{K}| \geq q + n$. □

We put $r := k - q - 2$. The following is a consequence of Lemma 5.1.

Corollary 5.2. *If there are thick planes, then* $r = |\mathcal{K} \setminus \alpha| \geq n - 2$*.*

5.2 The case where there are no thick planes

The case where there are no thick planes was treated in [7]. The following result was obtained there.

Proposition 5.3. *Suppose* $q \geq 3$ *and* $\mathcal{K}$ *is a cap of* $\mathrm{PG}(n, q)$ *satisfying property* $(*)$ *of Theorem 1.1 such that there are no thick planes. Then one of the following cases occurs:*

(1) $n = 2$ *and* $\mathcal{K}$ *is a set of three noncollinear points;*

(2) $n = 5$*,* $q = 3$ *and* $\mathcal{K}$ *is a Coxeter cap in* $\mathrm{PG}(5, 3)$*.*

If case (1) of Proposition 5.3 occurs, then the associated near hexagon $T_2^*(\mathcal{K})$ is a so-called Hamming near hexagon with q points per line. If case (2) of Proposition 5.3 occurs, then $T_5^*(\mathcal{K}) \cong \mathbb{E}_1$. We refer to [7] for more details.

The main tools in the proof of Proposition 5.3 in [7] are Lemma 4.5 and the Haemers-Mathon bound for the associated regular near hexagon $T_n^*(\mathcal{K})$. This Haemers-Mathon bound was used in [7] to prove that $n \leq 7$.

5.3 The case where $n \in \{3, 4, 5\}$ and there exist thick planes

The case where $n \in \{3, 4, 5\}$ and there exist thick planes was treated in [7]. The following result was obtained there.

Proposition 5.4. *Suppose $n \in \{3, 4, 5\}$ and $\mathcal{K}$ is a set of points of $\mathrm{PG}(n, q)$ satisfying property $(*)$ of Theorem 1.1 for which there exists at least one thick plane (and so q is even). Then one of the following cases occurs:*

(1) *$n = 3$ and $\mathcal{K}$ consists of the hyperoval $\alpha \cap \mathcal{K}$ in the unique thick plane α and one point not in α;*

(2) *$n = 4$ and $\mathcal{K}$ is the union of two hyperovals whose carrying (thick) planes α and α' meet in a point belonging to both hyperovals.*

If case (1) of Proposition 5.4 occurs, then the associated near hexagon $T_3^*(\mathcal{K})$ is the direct product of a line of size q with the generalized quadrangle $T_2^*(\alpha \cap \mathcal{K})$. If case (2) of Proposition 5.4 occurs, then $T_4^*(\mathcal{K})$ is a so-called glued near hexagon of type $T_2^*(\alpha \cap \mathcal{K}) \otimes T_2^*(\alpha' \cap \mathcal{K})$. We refer to [7] for more details.

The proof of Proposition 5.4 in [7] was mainly geometric of nature. Also the case $n = 6$ was "excluded" in [7], but that proof seems not to be valid as it relies on the fact that k is congruent to 1 or 3 modulo $q + 1$, which is not necessarily true (see the final remark of Section 4). In Section 5.7, we therefore give a new proof for the nonexistence of this case.

5.4 Derivation of the upper bound $n \leq 8$

In [7], we used the Haemers-Mathon bound to derive an upper bound for n in case $T_n^*(\mathcal{K})$ is a regular near hexagon. In the general case, $T_n^*(\mathcal{K})$ is not necessarily a regular near hexagon. However, we can still use the generalized version of the Haemers-Mathon bound (Proposition 2.1) to obtain a slightly worse upper bound for n.

Lemma 5.5. *We have $n \leq 8$.*

Proof. From Lemma 4.5, we know that

$$\frac{q^n - 1}{q - 1} \leq \frac{1}{2}(q-1)(k-1)(k-2) + (k-1) = \frac{1}{2}st(t-1) + t.$$

By Lemma 4.4, we know that every two points at distance 2 have either 2 or $q+2 = s+3$ common neighbours. If x and y are two points of $T_n^*(\mathcal{K})$ at distance 3 from each other, and x, u, v, y is a shortest path between x and y, then xu is one of the $t+1$ lines through x, u is the unique point of xu at distance 2 from y and v is one of the neighbours of u and y. So, the number of shortest paths between x and y is bounded above by $(t+1)(s+3)$. From Proposition 2.1, we then know that

$$t \leq s^3 + (s+2)(s^2 - s + 1) = 2s^3 + s^2 - s + 2.$$

Hence,

$$\frac{(s+1)^n-1}{s} = \frac{q^n-1}{q-1} \le \frac{1}{2}s(2s^3+s^2-s+2)(2s^3+s^2-s+1)+2s^3+s^2-s+2$$
$$= 2s^7+2s^6-\frac{3}{2}s^5+2s^4+4s^3-\frac{1}{2}s^2+2.$$

If $n \ge 9$, then

$$s^8+9s^7+36s^6+84s^5+126s^4+126s^3+84s^2+36s+9 = \frac{(s+1)^9-1}{s}$$
$$\le \frac{(s+1)^n-1}{s} \le 2s^7+2s^6-\frac{3}{2}s^5+2s^4+4s^3-\frac{1}{2}s^2+2,$$

an obvious contradiction. Hence, $n \le 8$. □

5.5 A divisibility condition and some consequences

The near hexagon $T_n^*(\mathcal{K})$ has order $(s,t)=(q-1,k-1)$ and contains q^{n+1} points. The divisibility condition mentioned in Proposition 2.2 yields

$$\frac{(q-1)^5q^{n+1}}{q^2(q-2)(q^2-2q+2)+(q-1)(q-2)q^2t+q^{n+1}} \in \mathbb{N},$$

$$\frac{(q-1)^5q^{n-1}}{(q-2)(q^2-2q+2)+1+(q-1)(q-2)t+q^{n-1}-1} \in \mathbb{N},$$
$$\frac{(q-1)^4q^{n-1}}{q^2-3q+3+(q-2)t+q^{n-2}+q^{n-3}+\cdots+1} \in \mathbb{N}. \tag{1}$$

We denote the denominator in the last fraction by D.

Lemma 5.6. *For $n \in \{6,7,8\}$, we have $k \equiv 3 \pmod{q}$.*

Proof. Suppose that this is not the case. Then $k \equiv 2 \pmod{q}$ by Lemma 4.6(2), or equivalently $t \equiv 1 \pmod{q}$. By Lemma 5.1 and Proposition 5.3, we know that $q \ge 4$ is even. We have $\gcd(q,D)=\gcd(q,4-2t)=\gcd(q,2)=2$. As $q \ge 4$ is a power of 2, D is even and $\frac{D}{2}$ is odd. This implies by (1) that D is a divisor of $2(q-1)^4$. For $n \ge 7$, this is impossible as we then have

$$D > q^5+q^4+q^3+2q^2-2q+4 > 2q^4-8q^3+12q^2-8q+2 = 2(q-1)^4.$$

Indeed, the second inequality is equivalent with $q^5-q^4+9q^3-10q^2+6q+2 = q^4(q-1)+q^2(9q-10)+6q+2>0$, which is always true. Suppose therefore that $n=6$. Then

$$D = q^4+q^3+2q^2-2q+4+(q-2)t > (q-1)^4.$$

As this a divisor of $2(q-1)^4$, we should have

$$q^4 + q^3 + 2q^2 - 2q + 4 + (q-2)t = 2(q-1)^4 = 2q^4 - 8q^3 + 12q^2 - 8q + 2,$$

i.e.

$$(q-2)t = q^4 - 9q^3 + 10q^2 - 6q - 2.$$

The right hand side is negative if $q = 4$. So, $q \geq 8$ and

$$t = q^3 - 7q^2 - 4q - 14 - \frac{30}{q-2}.$$

As $q \geq 8$ is a power of 2 and $q - 2 \mid 30$, we have $q \in \{8, 32\}$. If $q = 8$, then $t = 13$. If $q = 32$, then $t = 25457$. In each of these cases, we see that t is not congruent to 1 modulo q. □

Lemma 5.7. *Let $n \in \{6, 7, 8\}$. Then there exists a $\beta \in \mathbb{Z}$ such that $k = qn - 2q + 3 + \beta q(q-1)$.*

Proof. By Lemma 5.6, we know that $k \equiv 3 \pmod{q}$, and by Lemma 4.6(1), we know that $k \equiv n+1 \pmod{q-1}$. The claim then follows from the Chinese remainder theorem, and the facts that $qn - 2q + 3 \equiv 3 \pmod{q}$ and $qn - 2q + 3 \equiv n + 1 \pmod{q-1}$. □

5.6 Inequalities involving the number r and some consequences

The inequality (3) in the following lemma was already proved in Lemma 7.8 of [7]. However, we later also need the other inequality (2) whose proof requires that we repeat part of the original arguments. We therefore give a proof of both inequalities:

Lemma 5.8. *If there are thick planes, then*

$$(q-1)r^2 + (2q^2 - q + 1)r - 2q^2\frac{q^{n-2}-1}{q-1} \geq 0, \tag{2}$$

$$(q-1)r^2 + (q^2 + q + 1)r - q^2(q+2)\frac{q^{n-2}-1}{q-1} \leq 0. \tag{3}$$

Proof. Let β be a 3-dimensional subspace of $\mathrm{PG}(n+1, q)$ intersecting $\mathrm{PG}(n, q)$ in a thick plane α. By Lemma 4.3, $Q := \beta \setminus \alpha$ is a quad for which the induced subgeometry $\mathcal{S}_Q$ is isomorphic to the generalized quadrangle $T_2^*(\alpha \cap \mathcal{K})$. We denote by M the number of edges in the collinearity graph of $T_n^*(\mathcal{K})$ between points of $\Gamma_1(Q)$ and points of $\Gamma_2(Q)$.

By Lemma 4.2(1), there are r 3-dimensional subspaces in $\mathrm{PG}(n, q)$ through α containing exactly one point of $\mathcal{K} \setminus \alpha$. If β_1 is one of these subspaces, then by Proposition 2.8 all $q^4 - q^3$ points of $\langle \beta, \beta_1 \rangle \setminus (\mathrm{PG}(n, q) \cup \beta)$ are points of $\Gamma_1(Q)$. By Lemma 4.2(1), there are also $(\frac{q^{n-2}-1}{q-1} - r)$ 3-dimensional subspaces of $\mathrm{PG}(n, q)$ through α disjoint from $\mathcal{K} \setminus \alpha$. If β_2 is one of these subspaces, then by Proposition 2.8 all $q^4 - q^3$ points of $\langle \beta, \beta_2 \rangle \setminus (\mathrm{PG}(n, q) \cup \beta)$ are contained in $\Gamma_2(Q)$.

If $x \in \Gamma_2(Q)$ and $y \in \Gamma_1(Q) \cap \Gamma_1(x)$, then $\pi_Q(y)$ is a point of the ovoid $O_x = \Gamma_2(x) \cap Q$ of $\mathcal{S}_Q$. As $\mathcal{S}_Q$ is a generalized quadrangle of order $(q-1, q+1)$, we have $|O_x| = 1 + (q-1)(q+1) = q^2$. Note also that if $z \in O_x$, then $|\Gamma_1(x) \cap \Gamma_1(z)| \in \{2, q+2\}$ by Lemma 4.4. So, we have

$$\Big(\frac{q^{n-2}-1}{q-1} - r\Big) \cdot (q^4 - q^3) \cdot q^2 \cdot 2 \leq M \leq \Big(\frac{q^{n-2}-1}{q-1} - r\Big) \cdot (q^4 - q^3) \cdot q^2 \cdot (q+2). \quad (4)$$

By the above, we know that there are $|Q| \cdot s(t+1-(q+2)) = (q^4 - q^3)r$ points in $\Gamma_1(Q)$. Let y be one of these points. Every line L through y that lies entirely in $\Gamma_1(Q)$ is contained in a (necessarily unique) quad through $y\pi_Q(y)$ meeting Q in a line, namely the unique quad through y and $\pi_Q(z)$, where z is some point of $L \setminus \{y\}$. Note that every quad through $y\pi_Q(y)$ that intersects Q in a line through $\pi_Q(y)$ contains by Lemma 4.4 either 1 or $q+1$ lines through y that are contained in $\Gamma_1(Q)$. So, the total number of lines through y contained in $\Gamma_1(Q)$ lies in the interval $[q+2, (q+2)(q+1)]$. There is also a unique line through y meeting Q, namely $y\pi_Q(y)$. So, the total number of lines through y meeting $\Gamma_2(Q)$ lies in the interval $[t-(q+2)(q+1), t-(q+2)] = [r-(q+1)^2, r-1]$. Notice also that each line through y meeting $\Gamma_2(Q)$ contains precisely $q-1$ points of $\Gamma_2(Q)$ by Proposition 2.6. The number M of edges between $\Gamma_1(Q)$ and $\Gamma_2(Q)$ therefore also satisfies the following inequalities:

$$(q^4 - q^3)r(r-(q+1)^2)(q-1) \leq M \leq (q^4 - q^3)r(r-1)(q-1). \quad (5)$$

The lower bound for M obtained in (4) is at most the upper bound for M obtained in (5). The resulting inequality is equivalent with inequality (2). Similarly, the lower bound for M obtained in (5) is at most the upper bound for M obtained in (4). The resulting inequality is equivalent with inequality (3). $\square$

The following lemma was proved in Corollary 7.7 of [7]. We now provide a (partially) alternative argument based on the inequality (3).

Lemma 5.9. *If $n \geq 6$, then there are thick planes and $q \geq 8$ is even.*

Proof. By Lemma 5.1 and Proposition 5.3, there are thick planes and $q \geq 4$ is even. Suppose by way of contradiction that $q < 8$. Then $n \geq 6$ and $q = 4$. If β is a 3-dimensional subspace of $\mathrm{PG}(n+1, q)$ intersecting $\mathrm{PG}(n, q)$ in a thick plane α, then by Lemma 4.3 $Q := \beta \setminus \alpha$ is a quad on which the induced subgeometry $\mathcal{S}_Q$ is isomorphic to the generalized quadrangle $T_2^*(\alpha \cap \mathcal{K})$ mentioned in Proposition 2.15. As this generalized quadrangle has no rosettes of ovoids, we know from Propositions 2.5 and 2.7 that $T_n^*(\mathcal{K})$ has no points at distance 2 from Q (otherwise there exists a line meeting $\Gamma_1(Q)$ and $\Gamma_2(Q)$). As we have seen in the proof of Lemma 5.8, this implies that every 3-dimensional subspace of $\mathrm{PG}(n, q)$ through α contains a unique point of $\mathcal{K} \setminus \alpha$. So, $r = \frac{q^{n-2}-1}{q-1}$. Then inequality (3) in Lemma 5.8 implies that

$$0 \geq \frac{q^{n-2}-1}{q-1}(q^{n-2} - q^3 - q^2 + q),$$

which is impossible as $n - 2 \geq 4$. $\square$

5.7 The case $n = 6$ cannot occur

By Lemma 5.7, we have $k = 4q+3+\beta q(q-1)$, i.e. $r = 3q+1+\beta q(q-1)$ for some $\beta \in \mathbb{Z}$. By Lemma 5.9, we know that there are thick planes and $q \geq 8$ is even.

Lemma 5.10. *We have $q+1 \mid (\beta-1)(\beta-2)$.*

Proof. By Lemma 4.6(3), we have $q+1 \mid (k-1)(k-3)$, i.e. $(q+1) \mid ((4q+2)+\beta q(q-1))(4q+\beta q(q-1))$. This divisibility condition is equivalent with $(q+1) \mid (-2+2\beta)(-4+2\beta)$, i.e. with $(q+1) \mid (\beta-1)(\beta-2)$ as $q+1$ is odd. □

Lemma 5.11. *We have $\beta \geq 3$.*

Proof. As $r \geq n-2 = 4$ by Corollary 5.2, we have $0 \leq r-4 = (q-1)(3+\beta q)$, implying that $\beta \geq 0$ as $q \geq 8$. As $q+1 \geq 9$ is not a divisor of 2, we have $\beta \neq 0$ by Lemma 5.10.

Suppose $\beta = 1$. Then $r = q^2+2q+1$. We then have

$$(q-1)(q^2+2q+1)^2 + (2q^2-q+1)(q^2+2q+1) - 2q^2(q^3+q^2+q+1)$$
$$= -q^5+3q^4+3q^3-3q^2-2q = -q^3((q-4)(q+1)+1) - 3q^2 - 2q < 0,$$

in contradiction with inequality (2) in Lemma 5.8.

Suppose $\beta = 2$. Then $r = 2q^2+q+1$ and $t = 2q^2+2q+2$. The divisibility condition (1) from Section 5.5 becomes

$$\frac{(q-1)^4 q^5}{q^2-3q+3+2(q-2)(q^2+q+1)+q^4+q^3+q^2+q+1} \in \mathbb{N},$$

i.e.

$$\frac{(q-1)^3 q^4}{(q+2)^2} \in \mathbb{N}.$$

The remainder of the division of $(x-1)^3 x^4 \in \mathbb{Z}[x]$ by $x+2 \in \mathbb{Z}[x]$ is equal to $(-3)^3(-2)^4 = -432$. The only power q of 2 that is bigger than 4 for which $q+2$ divides $(q-1)^3q^4$, or equivalently 432, is equal to 16. But if $q = 16$, then $(q+2)^2$ is not a divisor of $(q-1)^3q^4$. □

By inequality (3), we have $r \leq r^*$ where

$$r^* := \frac{-q^2-q-1+\sqrt{4q^7+8q^6+q^4-2q^3-5q^2+2q+1}}{2(q-1)}.$$

If we put $\eta := 4q^7+8q^6+q^4-2q^3-5q^2+2q+1$, then from $r \leq r^*$, we deduce that

$$\sqrt{\eta} \geq 2(q-1)(3q+1) + \beta \cdot 2q(q-1)^2 + q^2+q+1 = 7q^2-3q-1+\beta \cdot 2q(q-1)^2.$$

Putting $q = u^2$, we thus find

$$4u^{14}+8u^{12}+u^8-2u^6-5u^4+2u^2+1 \geq (7u^4-3u^2-1+\beta \cdot 2u^2(u^2-1)^2)^2.$$

As

$$(7u^4-3u^2-1+(u+1)2u^2(u^2-1)^2)^2 = 4u^{14}+8u^{13}-12u^{12}-4u^{11}+36u^{10}-20u^9-11u^8$$

$$+16u^7-6u^6+4u^5-5u^4-4u^3+2u^2+1 > 4u^{14}+8u^{12}+u^8-2u^6-5u^4+2u^2+1$$

for $u \geq \sqrt{8}$, we have $\beta < u+1$. Taking into account that $\beta \geq 3$, we thus have

$$0 < (\beta-1)(\beta-2) < u(u-1) = q-\sqrt{q} < q,$$

in contradiction with the fact that $q+1$ is a divisor of $(\beta-1)(\beta-2)$.

5.8 The case $n=7$ cannot occur

By Lemma 5.7, we have $k = 5q+3+\beta q(q-1)$ and $r = 4q+1+\beta q(q-1)$ for some $\beta \in \mathbb{Z}$. By Lemma 5.9, we know that there are thick planes and $q \geq 8$ is even.

Lemma 5.12. *We have $q+1 \mid (\beta-2)^2$.*

Proof. By Lemma 4.6(3), we have $q+1 \mid (k-2)^2$, i.e. $(q+1) \mid (5q+1+\beta q(q-1))^2$. This divisibility condition is equivalent with $(q+1) \mid 4(\beta-2)^2$, i.e. with $(q+1) \mid (\beta-2)^2$ as $q+1$ is odd. $\square$

As $t = 5q+2+\beta q(q-1)$, the divisibility condition (1) from Section 5.5 implies that

$$\frac{(q-1)^3 q^5}{q^3+2q^2+3q+10+\beta(q-2)} \in \mathbb{N}.$$

Lemma 5.13. *We have $\beta \geq 2$.*

Proof. As $r \geq n-2 = 5$ by Corollary 5.2, we have $0 \leq r-5 = (q-1)(4+\beta q)$, implying that $\beta \geq 0$ as $q \geq 8$. As $q+1 \geq 9$ is a divisor of $(\beta-2)^2$, we also have $\beta \notin \{0,1\}$. $\square$

By inequality (3), we have $r \leq r^*$ where

$$r^* := \frac{-q^2-q-1+\sqrt{4q^8+8q^7+q^4-2q^3-5q^2+2q+1}}{2(q-1)}.$$

If we put $\eta := 4q^8+8q^7+q^4-2q^3-5q^2+2q+1$, then from $r \leq r^*$, we deduce that

$$\sqrt{\eta} \geq 2(q-1)(4q+1)+\beta\cdot 2q(q-1)^2+q^2+q+1 = 9q^2-5q-1+\beta\cdot 2q(q-1)^2.$$

As

$$(9q^2-5q-1+(q+4)2q(q-1)^2)^2 = 4q^8+16q^7-4q^6-28q^5+45q^4-38q^3+19q^2-6q+1$$

$$> 4q^8+8q^7+q^4-2q^3-5q^2+2q+1$$

for $q \geq 8$, we have $\beta \leq q+3$.

Suppose now that $q = 8$. We then know that

$$\frac{11239424}{674 + 6\beta} \in \mathbb{N}$$

where $2 \leq \beta \leq q+3 = 11$. This only turns out to be the case for $\beta = 2$. But then $r = 145$ and one can verify that the inequality (2) in Lemma 5.8 is not satisfied. So, $q \geq 16$.

As $q \geq 16$, we have

$$(q-1)^3 q^5 = q^8 - 3q^7 + 3q^6 - q^5 < q^8 - 2q^7 - 3q^6 - 14q^5 - 24q^4 = (q^3+2q^2+5q+6)(q^5-4q^4),$$

$$(q-1)^3 q^5 = q^8 - 3q^7 + 3q^6 - q^5 > q^8 - 3q^7 - 14q^6 - 20q^5 - 24q^4 = (q^3+3q^2+4q+4)(q^5-6q^4).$$

Under the assumptions that $2 \leq \beta \leq q+3$ and $q \geq 16$, we thus have

$$\frac{(q-1)^3 q^5}{q^3 + 2q^2 + 3q + 10 + \beta(q-2)} \leq \frac{(q-1)^3 q^5}{q^3 + 2q^2 + 5q + 6} < q^5 - 4q^4$$

and

$$\frac{(q-1)^3 q^5}{q^3 + 2q^2 + 3q + 10 + \beta(q-2)} \geq \frac{(q-1)^3 q^5}{q^3 + 3q^2 + 4q + 4} > q^5 - 6q^4.$$

The highest power of 2 dividing $\frac{(q-1)^3 q^5}{q^3+2q^2+3q+10+\beta(q-2)}$ is therefore at most q^4. So, the highest power of 2 dividing $q^3 + 2q^2 + 3q + 10 + \beta(q-2)$ is at least q, implying that $10 - 2\beta$ is a multiple of q.

As $2 \leq \beta \leq q+3$, we have $4 - 2q \leq 10 - 2\beta \leq 6$. In view of the fact that $10 - 2\beta$ is a multiple of $q \geq 16$, we thus have $10 - 2\beta \in \{0, -q\}$, i.e. $\beta \in \{5, \frac{q}{2} + 5\}$.

We also need that $q + 1 \mid (\beta - 2)^2$ by Lemma 5.12. For $\beta = 5$, this would mean that $q + 1 \mid 9$, which is impossible as $q \geq 16$. For $\beta = \frac{q}{2} + 5$, this would mean that $(q+1) \mid (q+6)^2$, or equivalently that $(q+1) \mid 25$, which is again impossible. So, all together we have shown that the case $n = 7$ cannot occur.

5.9 The case $n = 8$ cannot occur

By Lemma 5.7, we have $k = 6q + 3 + \beta q(q-1)$ and $r = 5q + 1 + \beta q(q-1)$ for some $\beta \in \mathbb{Z}$. By Lemma 5.9, we know that there are thick planes and $q \geq 8$ is even.

Lemma 5.14. *We have $\beta = 6 + q\gamma$ for a certain $\gamma \in \mathbb{Z}$.*

Proof. From the divisibility condition (1) in Section 5.5, we know that

$$\frac{(q-1)^4 q^7}{q^2 - 3q + 3 + (q-2)(6q + 2 + \beta q(q-1)) + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \in \mathbb{N},$$

$$\frac{(q-1)^4 q^7}{(q-1)q(q^4 + 2q^3 + 3q^2 + 4q + 12) + \beta q(q-1)(q-2)} \in \mathbb{N},$$

$$\frac{(q-1)^3q^6}{q^4+2q^3+3q^2+4q+12+\beta(q-2)} \in \mathbb{N}.$$

Put $D = q^4+2q^3+3q^2+4q+12+\beta(q-2)$. As $\gcd((q-1)^3, q^6) = 1$, we have $D = \gcd(D, (q-1)^3q^6) = \gcd(D, (q-1)^3) \cdot \gcd(D, q^6)$. As $D > q(q-1)^3$ and $\gcd(D, (q-1)^3) \leq (q-1)^3$, we have $\gcd(D, q^6) > q$ and hence $\gcd(D, q^6) \geq 2q$ as q is a power of 2. So, $2q \mid (q^4+2q^3+3q^2+4q+12+\beta q-2\beta)$, i.e. $2q \mid (12+\beta q-2\beta)$. As $4 \mid q$, we have $2 \mid \beta$ and so the fact that $2q$ divides $12+\beta q-2\beta$ implies that $2q$ divides $12-2\beta$, or equivalently that q divides $\beta-6$. We thus have that $\beta = 6+q\gamma$ for a certain $\gamma \in \mathbb{Z}$. $\square$

We thus have $r = 5q+1+(6+q\gamma)q(q-1) = 6q^2-q+1+q^2(q-1)\gamma$ and $k = 6q^2+3+q^2(q-1)\gamma$.

Lemma 5.15. *We have* $(q+1) \mid (\gamma-3)(\gamma-4)$.

Proof. By Lemma 4.6(3), we have $(q+1) \mid (k-1)(k-3)$, i.e. $(q+1) \mid (6q^2+2+\gamma q^2(q-1))(6q^2+\gamma q^2(q-1))$. This divisibility condition is equivalent with $(q+1) \mid (8-2\gamma)(6-2\gamma)$, i.e. with $(q+1) \mid (\gamma-3)(\gamma-4)$ as $q+1$ is odd. $\square$

Lemma 5.16. *We have* $\gamma \geq 5$.

Proof. As $r \geq n-2 = 6$ by Corollary 5.2, we have $0 \leq r-6 = (q-1)(6q+5+q^2\gamma)$, implying that $\gamma \geq 0$ as $q \geq 8$. As $q+1$ is not a divisor of 2, 6 and 12, we have $\gamma \notin \{0,1,2\}$ by Lemma 5.15.

Suppose $\gamma = 3$. Then $t = 3q^3+3q^2+2$. The divisibility condition (1) from Section 5.5 becomes

$$\frac{(q-1)^4q^7}{q^2-3q+3+(q-2)(3q^3+3q^2+2)+q^6+q^5+q^4+q^3+q^2+q+1} \in \mathbb{N},$$

i.e.

$$\frac{(q-1)^3q^5}{q^3+2q^2+6q+4} \in \mathbb{N}.$$

As $q \geq 8$ is a power of 2, $\gcd(q^3+2q^2+6q+4, q^5) = 4$, implying that q^3+2q^2+6q+4 is a divisor of $4(q-1)^3$. The highest power of 2 dividing q^3+2q^2+6q+4 and $4(q-1)^3$ is in both cases equal to 4. As $4(q^3+2q^2+6q+4) > 4(q-1)^3$, we have that $4(q-1)^3$ is equal to either q^3+2q^2+6q+4 or $3(q^3+2q^2+6q+4)$. The former equation is equivalent with $3q^3-14q^2+6q-8 = 0$ while the latter equation is equivalent with $q^3-18q^2-6q-16$. In any case, there are no solutions for q.

Suppose $\gamma = 4$. Then $t = 4q^3+2q^2+2$. The divisibility condition (1) from Section 5.5 becomes

$$\frac{(q-1)^4q^7}{q^2-3q+3+(q-2)(4q^3+2q^2+2)+q^6+q^5+q^4+q^3+q^2+q+1} \in \mathbb{N},$$

i.e.

$$\frac{(q-1)^3q^5}{q^3+2q^2+7q+2} \in \mathbb{N}.$$

As $q \geq 8$ is a power of 2, $\gcd(q^3 + 2q^2 + 7q + 2, q^5) = 2$ and so $q^3 + 2q^2 + 7q + 2$ is a divisor of $2(q-1)^3$, implying that $q^3 + 2q^2 + 7q + 2 = 2(q-1)^3$, i.e. $q^3 - 8q^2 - q - 4 = 0$, a contradiction. $\square$

By inequality (3), we have $r \leq r^*$ where

$$r^* := \frac{-q^2 - q - 1 + \sqrt{4q^9 + 8q^8 + q^4 - 2q^3 - 5q^2 + 2q + 1}}{2(q-1)}.$$

If we put $\eta := 4q^9 + 8q^8 + q^4 - 2q^3 - 5q^2 + 2q + 1$, then from $r \leq r^*$, we deduce that

$$\sqrt{\eta} \geq 2(q-1)(6q^2 - q + 1) + 2q^2(q-1)^2\gamma + q^2 + q + 1 = 12q^3 - 13q^2 + 5q - 1 + 2q^2(q-1)^2\gamma.$$

Putting $q = u^2$, we thus find

$$4u^{18} + 8u^{16} + u^8 - 2u^6 - 5u^4 + 2u^2 + 1 \geq (12u^6 - 13u^4 + 5u^2 - 1 + 2u^4(u^2-1)^2\gamma)^2.$$

As

$$\begin{aligned}(12u^6 - 13u^4 + 5u^2 - 1 + 2u^4(u^2-1)^2(u+1))^2 &= 4u^{18} + 8u^{17} - 12u^{16} + 16u^{15} + 56u^{14} - 100u^{13} \\ &+4u^{12} + 140u^{11} - 152u^{10} - 88u^9 + 197u^8 + 28u^7 + 126u^6 - 4u^5 + 47u^4 - 10u^2 + 1 \\ &> 4u^{18} + 8u^{16} + u^8 - 2u^6 - 5u^4 + 2u^2 + 1\end{aligned}$$

for $u \geq \sqrt{8}$, we have $\gamma < u + 1$. Taking into account that $\gamma \geq 5$, we thus have

$$0 < (\gamma - 3)(\gamma - 4) < (u-2)(u-3) = (\sqrt{q} - 2)(\sqrt{q} - 3) < q,$$

in contradiction with the fact that $q + 1$ is a divisor of $(\gamma - 3)(\gamma - 4)$.

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References

[1] A. E. Brouwer. The uniqueness of the near hexagon on 729 points. *Combinatorica* 2 (1982), 333–340.

[2] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[3] H. S. M. Coxeter. Twelve points in $PG(5, 3)$ with 95040 self-transformations. *Proc. Roy. Soc. London. Ser. A* 247 (1958), 279–293.

[4] B. De Bruyn. *Near polygons.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[5] B. De Bruyn. On the adjacency algebras of near hexagons with an order. *Graphs Combin.* 33 (2017), 1219–1230.

[6] B. De Bruyn. On tetrahedrally closed line systems and a generalization of the Haemers-Roos inequality. *Linear Algebra Appl.* 600 (2020), 130–147.

[7] B. De Bruyn and F. De Clerck. On linear representations of near hexagons. *European J. Combin.* 20 (1999), 45–60.

[8] M. J. E. Golay. Notes on digital coding. *Proc. I.R.E.* 37 (1949), 657.

[9] W. Haemers and R. Mathon. An inequality for near hexagons. Unpublished manuscript (1979).

[10] W. Haemers and C. Roos. An inequality for generalized hexagons. *Geom. Dedicata* 10 (1981), 219–222.

[11] R. Mathon. On primitive association schemes with three classes. Unpublished manuscript.

[12] A. Neumaier. Krein conditions and near polygons. *J. Combin. Theory Ser. A* 54 (1990), 201–209.

[13] S. E. Payne. The generalized quadrangle with $(s,t) = (3,5)$. Proceedings of the Twenty-first Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1990). *Congr. Numer.* 77 (1990), 5–29.

[14] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society, Zürich, 2009.

[15] E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.