Colouring graphs with no induced six-vertex path or diamond*

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In loving memory of Johan Robaey

Abstract

The diamond is the graph obtained by removing an edge from the complete graph on 4 vertices. A graph is (P_6, diamond)-free if it contains no induced subgraph isomorphic to a six-vertex path or a diamond. In this paper we show that the chromatic number of a (P_6, diamond)-free graph G is no larger than the maximum of 6 and the clique number of G. We do this by reducing the problem to imperfect (P_6, diamond)-free graphs via the Strong Perfect Graph Theorem, dividing the imperfect graphs into several cases, and giving a proper colouring for each case. We also show that there is exactly one 6-vertex-critical (P_6, diamond, K_6)-free graph. Together with the Lovász theta function, this gives a polynomial time algorithm to compute the chromatic number of (P_6, diamond)-free graphs.

Keywords. Graph colouring; χ-boundedness; strong perfect graph theorem; Lovász theta function; polynomial-time algorithms.

1 Introduction

All graphs in this paper are finite and simple. For general graph theory notation we follow [3]. A *q-colouring* of a graph G assigns a colour from a colour set $\{1, \ldots, q\}$ to each vertex of G such that adjacent vertices are assigned different colours. We say that a graph G is *q-colourable* if G admits a q-colouring.

Graph colouring was originally motivated by colouring maps. The famous Four Colour Theorem (which was proven by Appel and Haken [1]) says that every planar graph is 4-colourable. So every planar map can be coloured with 4 colours such that neighbouring countries have different colours. Many other real-life problems can be modelled by graph colouring. Here we give an example of scheduling. Assume that we have a set of jobs. Each job consumes an arbitrary processor for a specific period of time, while each processor can execute at most one job at the same time. Let G be a graph such that the vertices are the jobs, and two jobs are

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adjacent if their time periods overlap. Now the jobs can be done with q processors if and only if G is q-colourable. There are many other applications of graph colouring, such as memory allocation [2, 4], aircraft scheduling [25] and so on [14, 15].

The problem of deciding if a graph is *q-colourable* is called the *q-colouring problem*. This decision problem is NP-complete for general graphs for every $q \geq 3$. However, there exist polynomial time algorithms when the input graphs are restricted to certain graph classes. For example, Chudnovsky, Spirkl, and Zhong [11] recently showed that the 4-colouring problem can be solved in polynomial time for the class of P_6-free graphs. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which G is q-colourable.

Let P_n, C_n and K_n denote the path, cycle and complete graph on n vertices, respectively. The *diamond* is the graph obtained from K_4 by removing an edge. For two graphs G and H, we use $G+H$ to denote the *disjoint union* of G and H, and $G \vee H$ to denote the graph obtained from the disjoint union of G and H by adding an edge between every vertex in G and every vertex in H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. A *hole* is an induced cycle on 4 or more vertices. An *antihole* is the complement of a hole. A hole or antihole is *odd* or *even* if it has an odd or even number of vertices. We say that a graph G *contains* a graph H if an induced subgraph of G is isomorphic to H. A graph G is *H-free* if it does not contain H. For a family $\mathcal{H}$ of graphs, G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. The graphs in $\mathcal{H}$ are the *forbidden induced subgraphs* of the family of $\mathcal{H}$-free graphs. We write $(H_1, \ldots, H_n)$-free instead of $\{H_1, \ldots, H_n\}$-free. A *clique* is a vertex set whose elements are pairwise adjacent. A vertex set is *stable* if its elements are pairwise nonadjacent, and *nonstable* otherwise. The *clique number* of G, denoted by $\omega(G)$, is the size of a largest clique in G. Obviously, $\chi(G) \geq \omega(G)$ for any graph G.

A graph family $\mathcal{G}$ is *hereditary* if $G \in \mathcal{G}$ implies that every induced subgraph of G belongs to $\mathcal{G}$. Obviously, $\mathcal{G}$ is hereditary if and only if $\mathcal{G}$ is the class of $\mathcal{H}$-free graphs for some $\mathcal{H}$. A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G, and *imperfect* otherwise. As a generalisation of perfect graphs, Gyárfás [20] introduced the χ-bounded graph families. A hereditary graph family $\mathcal{G}$ is *χ-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. The function f is called a *χ-binding function*. Chudnovsky, Robertson, Seymour and Thomas [10] characterized the family of perfect graphs by forbidden induced subgraphs:

Theorem 1 ([10])**.** *A graph is perfect if and only if it does not contain an odd hole or an odd antihole as an induced subgraph.*

In other words, the family of (odd hole, odd antihole)-free graphs is χ-bounded, and its χ-binding function is the identity function. Based on this theorem, researchers studied various graph families with two forbidden induced subgraphs and found several families that have a linear χ-binding function. Note that every graph family mentioned in this paragraph forbids odd holes and odd antiholes on 7 or more vertices. It then follows from Theorem 1 that every graph in these graph families is either perfect or contains a C_5. In particular, Gaspers and Huang [16] showed that $\chi(G) \leq \frac{3}{2}\omega(G)$ for every (P_6,C_4)-free graph G. Karthick and Maffray [22] improved the χ-binding function of (P_6,C_4)-free graphs to $\frac{5}{4}\omega(G)$. The graph $P_4 \vee K_1$ is called a *gem*, and a *co-gem* is the complement of a gem. Cameron, Huang and Merkel [6] showed that $\chi(G) \leq \lfloor\frac{3}{2}\omega(G)\rfloor$ for every (P_5, gem)-free graph G. Chudnovsky, Karthick, Maceli, and Maffray [9] improved the χ-binding function of (P_5, gem)-free graphs to $\lceil\frac{5}{4}\omega(G)\rceil$. Karthick and Maffray [21] showed that $\chi(G) \leq \lceil\frac{5}{4}\omega(G)\rceil$ for every (gem, co-gem)-free graph G, and in [23] they showed that every (P_5, diamond)-free graph G satisfies $\chi(G) \leq \omega(G)+1$. For the family of (P_6, diamond)-free graphs, Karthick and Mishra [24] showed that every (P_6, diamond)-free graph G satisfies $\chi(G) \leq 2\omega(G)+5$. In the same paper, they proved that every (P_6, diamond, K_4)-free graph is 6-colourable. Finally, Cameron, Huang, and Merkel [7] improved the χ-binding function of (P_6, diamond)-free graphs to $\omega(G)+3$.

Our Contributions

In this paper, we prove that every (P_6, diamond)-free graph G satisfies $\chi(G) \leq \max\{6, \omega(G)\}$ (cf. Theorem 5 in Section 4). We do this by reducing the problem to imperfect (P_6, diamond)-free graphs via the Strong Perfect Graph Theorem, dividing the imperfect graphs into several cases, and giving a proper colouring for each case.

Furthermore, we prove that the chromatic number of (P_6, diamond)-free graphs can be determined in polynomial time (cf. Theorem 7 in Section 5). In particular, we show that there is exactly one 6-vertex-critical (P_6, diamond, K_6)-free graph. Together with the Lovász theta function, this gives a polynomial time algorithm to compute the chromatic number of (P_6, diamond)-free graphs. Note that Dabrowski, Dross and Paulusma [13] proved the following dichotomy for computing the chromatic number of (diamond,H)-free graphs when H has at most 5 vertices: they proved that this problem is NP-complete when H contains a claw or a cycle, and polynomial time solvable otherwise. Our result thus generalises the polynomial time solvability of computing the chromatic number of (P_5,diamond)-free graphs.

Our results are an improvement of the result of Cameron, Huang and Merkel [7], answer an open question from [7], and are a natural next step of [13]. We believe that our proof technique for polynomial time solvability may also be useful for other graph families (see Section 6).

The remainder of the paper is organised as follows. We present some preliminaries in Section 2 and show some structural properties of imperfect (P_6, diamond)-free graphs in Section 3. We prove the χ-bound in Section 4 and prove that the chromatic number can be determined in polynomial time in Section 5. We end with some open problems in Section 6.

2 Preliminaries

Let $G = (V, E)$ be a graph. A *neighbour* of a vertex v is a vertex adjacent to v. The *neighbourhood* of a vertex v, denoted by $N_G(v)$, is the set of neighbours of v. The *degree* of a vertex v, denoted by $d(v)$, is the number of neighbours of v. We denote the minimum degree of the vertices of G by $\delta(G)$. For $X \subseteq V$, let $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$. For $x \in V$ (or $X \subseteq V$) and $S \subseteq V$, let $N_S(x) = N_G(x) \cap S$ (or $N_S(X) = N_G(X) \cap S$). For $x \in V$ (or $X \subseteq V$) and $Y \subseteq V$, we say that x (or X) is *complete* (resp. *anti-complete*) to Y if x (or every vertex in X) is adjacent (resp. nonadjacent) to every vertex in Y. We denote the complement of G by $\overline{G}$. For $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S. We often write S for $G[S]$ if the context is clear. We say that S induces an H if $G[S]$ is isomorphic to H. A clique $K \subseteq V$ is a *clique cutset* if $G - K$ has more components than G. Two vertices $u, v \in V$ are *comparable* if they are nonadjacent, and either $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$. A component of a graph is *trivial* if it has only one vertex, and *nontrivial* otherwise. We say that the edges between two vertex sets X and Y form a *matching* if every vertex in X has at most one neighbour in Y, and every vertex in Y has at most one neighbour in X.

Grötschel, Lovász and Schrijver [19] showed that the chromatic number of a perfect graph can be computed in polynomial time. In that paper, the authors used the Lovász theta function:

$$\vartheta(G) := max\{\sum_{i,j=1}^{n} b_{ij} :$$
$$B = (b_{ij}) \text{ is positive semidefinite with trace at most 1, and } b_{ij} = 0 \text{ if } ij \in E\}.$$

The Lovász theta function satisfies that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$ for any graph G, and can be calculated in polynomial time [19].

A graph G is *k-vertex-critical* if $\chi(G) = k$, and every proper induced subgraph of G has chromatic number smaller than k. The following properties of k-vertex-critical graphs are well-known.

Lemma 1 ([3])**.** *If G is a k-vertex-critical graph, then G is connected, has no clique cutsets or comparable vertices, and $\delta(G) \geq k-1$.*

It is easy to see that G is not $(k-1)$-colourable if and only if G contains a k-vertex-critical graph. This simple observation has an important algorithmic implication, the proof of which can be found in [5].

Theorem 2 (Folklore)**.** *If a hereditary graph family $\mathcal{G}$ has a finite number of k-vertex-critical graphs, then the $(k-1)$-colouring problem can be solved in polynomial time for $\mathcal{G}$ by simply testing if the input graph contains any of these k-vertex-critical graphs as induced subgraph.*

The concept of k-vertex-critical graphs is also important in the context of certifying algorithms [26]. An algorithm is *certifying* if, along with the answer given by the algorithm, it also gives a certificate which allows to verify in polynomial time that the output of the algorithm is indeed correct. In case of the k-colouring problem, a canonical certificate for yes-instances would be a proper k-colouring of the graph while a canonical certificate for no-instances would be a $(k+1)$-vertex-critical graph.

We will also use the following two theorems in our proof:

Theorem 3 ([11])**.** *The 4-colouring problem can be solved in polynomial time for the class of P_6-free graphs.*

Theorem 4 ([27])**.** *Let G be a (P_6, K_3)-free graph with no comparable vertices. Then G is 4-colourable. Furthermore, G is not 3-colourable if and only if it contains the Grötzsch graph as an induced subgraph and is an induced subgraph of the 16-vertex Clebsch graph. (See Figure 1 for drawings of the Clebsch graph and the Grötzsch graph.)*

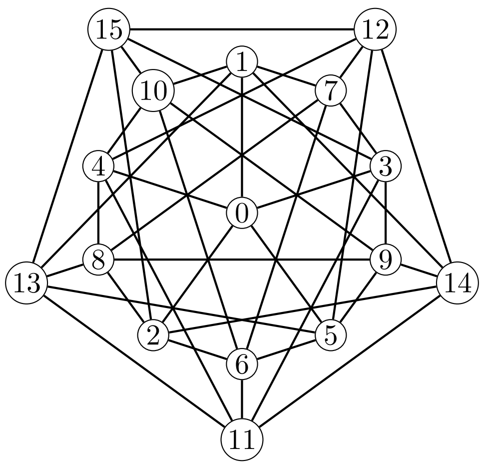

Figure 1: The Clebsch graph. The 11 vertices with indices $0, \ldots, 10$ induce the Grötzsch graph.

3 The Structure of Imperfect (P_6, diamond)-Free Graphs

In this section we study the structure of imperfect (P_6, diamond)-free graphs. By Theorem 1, every imperfect (P_6, diamond)-free graph contains a C_5. Let $G = (V, E)$ be an imperfect (P_6, diamond)-free graph. We follow the notation of [7] and partition V into the following subsets:

Let $Q=\{v_1, v_2, v_3, v_4, v_5\}$ induce a C_5 in G with edges v_iv_{i+1} for $i=1,\ldots,5$, with all indices modulo 5.

$A_i=\{v \in V\backslash Q : N_Q(v) = \{v_i\}\}$,

$B_{i,i+1}=\{v \in V\backslash Q : N_Q(v) = \{v_i, v_{i+1}\}\}$,

$C_{i,i+2}=\{v \in V\backslash Q : N_Q(v) = \{v_i, v_{i+2}\}\}$,

$F_i=\{v \in V\backslash Q : N_Q(v) = \{v_i, v_{i-2}, v_{i+2}\}\}$,

$Z=\{v \in V\backslash Q : N_Q(v) = \emptyset\}$.

Let $A=\bigcup_{i=1}^5 A_i$, $B=\bigcup_{i=1}^5 B_{i,i+1}$, $C=\bigcup_{i=1}^5 C_{i,i+2}$, $F=\bigcup_{i=1}^5 F_i$. Since G is diamond-free, any vertex in $V\backslash Q$ cannot be adjacent to three sequential vertices v_i, v_{i+1}, v_{i+2} in Q, then $V=Q \cup A \cup B \cup C \cup F \cup Z$.

In [7] the following 21 properties of these subsets are proved:

(P1) Each component of A_i is a clique.

(P2) The sets A_i and A_{i+1} are anti-complete.

(P3) The sets A_i and A_{i+2} are complete.

(P4) Each $B_{i,i+1}$ is a clique.

(P5) The set $B=B_{i,i+1} \cup B_{i+2,i+3}$ for some i.

(P6) The set $B_{i,i+1}$ is anti-complete to $A_i \cup A_{i+1}$.

(P7) The set $B_{i,i+1}$ is complete to $A_{i-1} \cup A_{i+2}$.

(P8) Each $C_{i,i+2}$ is a stable set.

(P9) Each vertex in $C_{i,i+2}$ is either complete or anti-complete to each component of A_i and A_{i+2}.

(P10) Each vertex in $C_{i,i+2}$ has at most one neighbour in each component of A_{i+1}, A_{i+3} and A_{i+4}.

(P11) Each vertex in $C_{i,i+2}$ is anti-complete to each nontrivial component of A_{i+1}.

(P12) The set $C_{i,i+2}$ is anti-complete to $B_{j,j+1}$ if $j \neq i+3$. Moreover each vertex in $C_{i,i+2}$ has at most one neighbour in $B_{i+3,i+4}$.

(P13) Each F_i has at most one vertex. Moreover, F is a stable set.

(P14) The set F_i is anti-complete to $A_{i+2} \cup A_{i+3}$.

(P15) Each vertex in F_i is either complete or anti-complete to each component of A_i.

(P16) Each vertex in F_i has at most one neighbour in each component of A_{i+1} and A_{i+4}.

(P17) The set F_i is anti-complete to $B_{j,j+1}$ if $j \neq i+2$ and complete to $B_{j,j+1}$ if $j = i+2$.

(P18) The set F_i is anti-complete to $C_{j,j+2}$ if $j \neq i-1$.

(P19) If A_i is not stable, then $A_{i+2} = A_{i+3} = B_{i+1,i+2} = B_{i-1,i-2} = \emptyset$.

(P20) If A_i is not empty, then each of $B_{i+1,i+2}$ and $B_{i-1,i-2}$ contains at most one vertex.

(P21) The set Z is anti-complete to $A \cup B$.

Now we prove some new properties which we will use in our proofs in Section 4 and Section 5:

(P22) If A_i, $B_{i+1,i+2}$ and $B_{i+3,i+4}$ are all nonempty, then $C_{i+1,i+3}$ (resp. $C_{i+2,i+4}$) is anti-complete to A_{i+2} (resp. A_{i+3}).

Proof. Suppose that $a_1 \in A_i$, $a_2 \in A_{i+2}$, $b_1 \in B_{i+1,i+2}$, $b_2 \in B_{i+3,i+4}$, $c \in C_{i+1,i+3}$ such that $a_2c \in E$. By (P3), $a_1a_2 \in E$. By (P6), $b_1a_2 \notin E$. By (P7), $a_1b_1, a_1b_2, a_2b_2 \in E$. By (P12), $b_1c, b_2c \notin E$. Then either $\{a_1, a_2, b_2, c\}$ induces a diamond or $\{b_1, a_1, a_2, c, v_{i+3}, v_{i+4}\}$ induces a P_6, depending on whether a_1 and c are adjacent. □

(P23) Suppose that $B_{i,i+1}$ is nonempty. Then $C_{i,i+2} \cup C_{i-2,i}$ (resp. $C_{i-1,i+1} \cup C_{i+1,i+3}$) is complete to A_{i-1} (resp. A_{i+2}). Moreover if A_{i-1} (resp. A_{i+2}) is nonempty, then $C_{i,i+2}$ and $C_{i-2,i}$ (resp. $C_{i-1,i+1}$ and $C_{i+1,i+3}$) are anti-complete.

Proof. Let $a \in A_{i-1}$, $b \in B_{i,i+1}$. Suppose there is a vertex $c_1 \in C_{i,i+2}$ such that $ac_1 \notin E$ or $c_2 \in C_{i-2,i}$ such that $ac_2 \notin E$. By (P12), $bc_1, bc_2 \notin E$. Then $\{c_1, v_{i+2}, v_{i-2}, v_{i-1}, a, b\}$ or $\{c_2, v_{i-2}, v_{i-1}, a, b, v_{i+1}\}$ induces a P_6. This proves the first part of the claim. Suppose that $c_1 \in C_{i,i+2}$ and $c_2 \in C_{i-2,i}$ are adjacent, then $\{a, c_1, c_2, v_i\}$ induces a diamond. □

(P24) Suppose that $A_i \cup A_{i+1}$ is nonempty. Then $B_{i,i+1}$ is complete to A_{i+3}. Moreover if $B_{i,i+1}$ contains two or more vertices, then A_{i+3} is empty.

Proof. By symmetry suppose that $a_1 \in A_i$, $a_2 \in A_{i+3}$, $b \in B_{i,i+1}$ such that $ba_2 \notin E$. By (P3), $a_1a_2 \in E$. By (P6), $ba_1 \notin E$. Then $\{b, v_i, a_1, a_2, v_{i+3}, v_{i+2}\}$ induces a P_6. This proves the first part of the claim. Suppose that $b_1, b_2 \in B_{i,i+1}$, $a \in A_{i+3}$. By (P4), $b_1b_2 \in E$. Then $\{v_i, b_1, b_2, a\}$ induces a diamond. □

(P25) If $B_{i,i+1}$ is nonempty, then $C_{i-2,i}$ (resp. $C_{i+1,i+3}$) is anti-complete to A_i (resp. A_{i+1}).

Proof. Suppose that $a \in A_i$, $b \in B_{i,i+1}$, $c \in C_{i-2,i}$ such that $ac \in E$. By (P6), $ab \notin E$. By (P12), $bc \notin E$. Then $\{a, c, v_{i-2}, v_{i+2}, v_{i+1}, b\}$ induces a P_6. □

(P26) If $B_{i,i+1}$ is nonempty, then Z is anti-complete to $C_{i+1,i+3} \cup C_{i+3,i}$.

Proof. By symmetry suppose that $b \in B_{i,i+1}$, $c \in C_{i+1,i+3}$ and $z \in Z$ such that $cz \in E$. By (P12), $bc \notin E$. By (P21), $bz \notin E$. Then $\{z, c, v_{i+3}, v_{i+4}, v_i, b\}$ induces a P_6. □

4 The χ-Bound of (P_6, diamond)-Free Graphs

In this section, we prove the following theorem.

Theorem 5. *Let G be a (P_6, diamond)-free graph, then $\chi(G) \le max\{6, \omega(G)\}$.*

This theorem is an improvement of the result of Cameron, Huang and Merkel [7] that $\chi(G) \le \omega(G) + 3$ for a (P_6, diamond)-free graph G. To prove Theorem 5, we use Theorem 6 below.

Theorem 6. *Let G be a connected imperfect (P_6, diamond)-free graph with no clique cutsets or comparable vertices. Then $\chi(G) \le max\{6, \omega(G)\}$.*

Proof of Theorem 5. If G is perfect, then $\chi(G) = \omega(G)$. If G is disconnected and is the union of components $\{H_1, \dots, H_n\}$, then $\chi(G)=max\{\chi(H_1), \dots, \chi(H_n)\}$ and $\omega(G)=max\{\omega(H_1), \dots, \omega(H_n)\}$. If G is connected and contains a clique cutset S, $G - S$ is the disjoint union of subgraphs $\{H_1, \dots, H_n\}$, then $\chi(G)=\max\{\chi(G[V(H_1) \cup S], \dots, \chi(G[V(H_n) \cup S]\}$ and $\omega(G)=\max\{\omega(G[V(H_1) \cup S], \dots, \omega(G[V(H_n) \cup S]\}$. If u and v are nonadjacent vertices in G such that $N(u) \subseteq N(v)$, then $\chi(G)=\chi(G - u)$ and $\omega(G)=\omega(G - u)$. Then the theorem follows from Theorem 6. □

Proof of Theorem 6. Let $G=(V,E)$. We use the partitioning of V from Section 3. Let $\omega := max\{6,\omega(G)\}$. In the following we give an ω-colouring for G. When we say that two vertex sets *do not conflict*, we mean that no edges between the two sets have two ends of the same colour.

Claim 1. *If $V\setminus Z$ has an ω-colouring such that one colour of $\{1,\ldots,\omega\}$ is not used in $C\cup F$, and F is coloured with at most 2 colours, then we can extend it to an ω-colouring of G.*

Proof. Let K be an arbitrary component of Z. By (P21), $N_G(K)\subseteq C\cup F$. If K contains exactly one vertex, then colour it with the colour not used in $C\cup F$. Now suppose that K contains at least two vertices.

It is proved in [7] that if K has no neighbours in C, then K is 3-colourable. (See Claim 1 in [7]. The conditions that G is connected and has no clique cutsets or comparable vertices are necessary.) Since F is coloured with at most 2 colours, we can colour K with the $\omega-2$ colours not used in F if K has no neighbour in C. Now suppose that K has a neighbour in C.

Let $c\in C_{i,i+2}$ be a neighbour of some vertex $z_1\in K$. If there is a vertex z_2 in K such that $z_1z_2\in E$ and $z_2c\notin E$, then $\{z_2,z_1,c,v_i,v_{i+4},v_{i+3}\}$ induces a P_6. So c is complete to K. Then $\{c\}\cup K$ is a clique since G is diamond-free. If there is more than one vertex in C complete to K, then $K\cup N_C(K)$ is a clique since K contains a K_2.

Suppose that c_1,c_2,c_3 are three neighbours of K in C, then $\{c_1,c_2,c_3\}$ is a clique. By (P8), we may assume that $c_1\in C_{i,i+2}$, $c_2\in C_{i+1,i+3}$, $c_3\in C_{i+2,i+4}\cup C_{i+3,i}$. Then either $\{c_1,c_2,c_3,v_{i+2}\}$ induces a diamond if $c_3\in C_{i+2,i+4}$, or $\{c_1,c_2,c_3,v_i\}$ induces a diamond if $c_3\in C_{i+3,i}$. So K has at most two neighbours in C.

Since K is a clique of size at least two, each vertex in $N_F(K)$ is either complete to K or adjacent to exactly one vertex in K, and no more than one vertex in F is complete to K by (P13). If $f\in F$ is complete to K, then $K\cup N_C(K)\cup\{f\}$ is a clique, and then $N_C(K)$ contains only one vertex by (P8) and (P18). Let $Y=\{v\in C\cup F : v \text{ is complete to } K\}$, then Y is a clique of size at most 2. Then each vertex z in K must have a neighbour in $F\setminus Y$, or else $Y\cup(K\setminus\{z\})$ is a clique cutset.

Now suppose that K contains three vertices z_1,z_2,z_3. Let $f_1,f_2,f_3\in F$ such that $z_if_j\in E$ if and only if $i=j$ for $i,j\in\{1,2,3\}$. By symmetry we may assume that $f_1\in F_i$, $f_2\in F_{i+1}$, $f_3\in F_{i+2}\cup F_{i+3}$. Then either $\{f_1,v_{i+3},f_2,z_2,z_3,f_3\}$ induces a P_6 if $f_3\in F_{i+2}$, or $\{f_1,v_i,f_3,z_3,z_2,f_2\}$ induces a P_6 if $f_3\in F_{i+3}$. So K contains exactly two vertices. Since K has at most 2 neighbours in C, and F is coloured with at most 2 colours, there are at least 2 colours left for K since $\omega\geq 6$.

Colour each component of Z this way, then Z and $V\setminus Z$ do not conflict. □

Now we will consider all cases and give an ω-colouring for each case. In the following, when we say that we colour a set with a certain colour, we mean that we colour each vertex in the set with that colour. When we say that we colour a clique or the disjoint union of some cliques with a set of colours, we mean that we colour each clique using the smallest colours available unless stated otherwise. For example, let A_1 be the disjoint union of three cliques K_1,K_2,K_3 of size 1,2,3 in order, when we say that we colour A_1 with colours in $\{1,3,4,5\}$, we mean that we colour K_1 with colour 1, K_2 with colours 1,3 and K_3 with colours 1,3,4.

By (P5), we may assume that $B=B_{2,3}\cup B_{4,5}$. In the following, we give a proper ω-colouring of $V\setminus Z$ such that one colour of $\{1,\ldots,\omega\}$ is not used in $C\cup F$, and F is coloured with at most 2 colours. Then by Claim 1, G is ω-colourable. We consider three cases: both $B_{2,3}$ and $B_{4,5}$ are nonempty, exactly one of $B_{2,3}$ and $B_{4,5}$ is nonempty, and B is empty.

Case 1. Both $B_{2,3}$ and $B_{4,5}$ are nonempty.

Case 1.1. Both $B_{2,3}$ and $B_{4,5}$ contain at least two vertices.

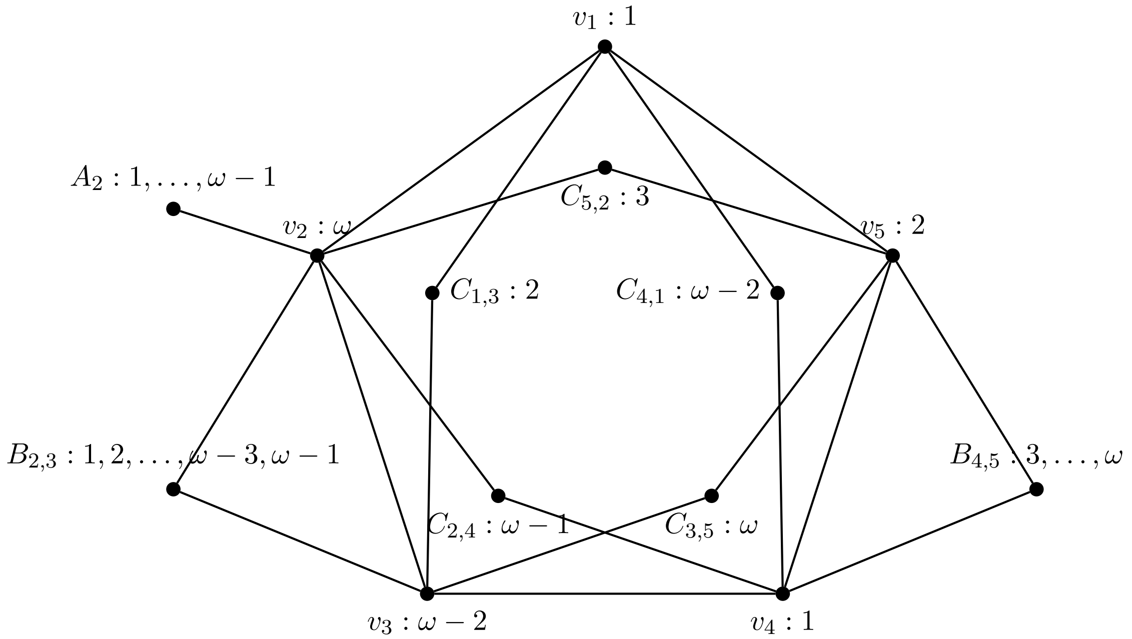

Figure 2: Diagram of the colouring of $Q \cup A \cup B \cup C$ in the proof of Case 1.1. Edges in $A \cup B \cup C$ are not drawn.

By (P20), $A_1 \cup A_3 \cup A_4$ is empty. By (P24), one of A_2 and A_5 is empty. By symmetry we assume that A_5 is empty. In the following we give a proper colouring of Q,B,A,C,F in turn. When we give a colouring of a vertex set, we prove that it does not conflict with the vertices we already coloured. So the whole colouring is proper. We give a diagram of the colouring of $Q \cup A \cup B \cup C$ in Figure 2.

- Colour $v_1, \ldots, v_5$ with colours $1, \omega, \omega - 2, 1, 2$ in order.

- By (P4), each of $B_{2,3}$ and $B_{4,5}$ is a clique of size at most $\omega - 2$. Colour $B_{4,5}$ with colours in $\{3, \ldots, \omega\}$. Exceptionally, when $B_{4,5}$ contains exactly $\omega - 3$ vertices, we use colours $\{3, 4, \ldots, \omega - 2, \omega\}$ instead of colours $\{3, \ldots, \omega - 1\}$.

- Since G is diamond-free, the edges between $B_{2,3}$ and $B_{4,5}$ form a matching. Sort the vertices in $B_{2,3}$ by the colour of their neighbours in $B_{4,5}$ in ascending order, putting the vertices with no neighbours in $B_{4,5}$ at the end of the queue. Then colour the vertices in the queue with colours $1, 2, \ldots, \omega - 3, \omega - 1$ in order. This ensures that for each edge between the two sets, its end in $B_{2,3}$ has a smaller colour than the end in $B_{4,5}$. So $B_{2,3}$ and $B_{4,5}$ do not conflict.

- Let K be an arbitrary component of A_2. By (P1), K is a clique of size at most $\omega - 1$. Then the edges between $B_{4,5}$ and K form a matching since G is diamond-free. Sort the vertices in K by the colour of their neighbours in $B_{4,5}$ in ascending order, putting the vertices with no neighbours at the end of the queue. Then colour the vertices in the queue with colours $1, \ldots, \omega - 1$ in order. Colour each component of A_2 this way. Then A_2 and $B_{4,5}$ do not conflict. By (P6), A_2 and $B_{2,3}$ do not conflict.

- By (P8), each $C_{i,i+2}$ can be coloured with one colour. Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $2, \omega - 1, \omega, \omega - 2, 3$ in order. By (P12), B and C do not conflict. By (P11), $C_{1,3}$ is anti-complete to each nontrivial component of A_2. (Note that each component of A_2 of size j is coloured with colours $1, \ldots, j$.) By (P25), $C_{5,2}$ is anti-complete to A_2.

Suppose that $a \in A_2$ with colour $\omega - 1$ is adjacent to $c \in C_{2,4}$. Then a belongs to a component K in A_2 of size $\omega - 1$. By (P9), $K \cup \{v_2, c\}$ is a clique of size $\omega + 1$.

Suppose that $a_1 \in A_2$ with colour $\omega - 2$ is adjacent to $c \in C_{4,1}$. Let b be the vertex of colour 3 in $B_{4,5}$. By the way we colour A_2, $a_1 b \notin E$, and there is a vertex $a_2 \in A_2$ adjacent to a_1, such that $a_2 b \notin E$. (Note that every vertex in A_2 which is adjacent to b is coloured with colour 1.) By (P10), $a_2 c \notin E$. Then $\{a_2, a_1, c, v_1, v_5, b\}$ induces a P_6. So C and A do not conflict.

- Colour $F_1, \ldots, F_5$ with colours $\omega - 1, \omega - 1, \omega, \omega - 1, \omega - 1$ in order. By (P13), this is a proper colouring of F. By (P18), F and C do not conflict. By (P17), $B_{2,3}$ and $F \setminus F_5$ do not conflict. Note that if $f \in F_5$ and $b \in B_{2,3}$ have the same colour $\omega - 1$, then $B_{2,3} \cup \{f, v_2, v_3\}$ is a clique of size $\omega + 1$ by (P17). So F and $B_{2,3}$ do not conflict. Similarly, F and $B_{4,5}$ do not conflict.

By (P14), F_4 and F_5 do not conflict with A_2. By (P15), F_2 and vertices in A_2 of colour $\omega - 1$ do not conflict, or else there is a clique of size $\omega + 1$ in $F_2 \cup A_2 \cup \{v_2\}$. Suppose that $a_1 \in A_2$ with colour $\omega - 1$ is adjacent to $f \in F_1$. Let b be the vertex of colour 3 in $B_{4,5}$. By the way we colour A_2, we have $a_1 b \notin E$, and there is a vertex $a_2 \in A_2$ adjacent to a_1 such that $a_2 b \notin E$. By (P16), $a_2 f \notin E$. Then $\{a_2, a_1, f, v_1, v_5, b\}$ induces a P_6. So A and F do not conflict. Now we have a proper colouring of $V \setminus Z$, so that the colour 1 is not used in $C \cup F$, and F is coloured with $\omega - 1$ and ω.

Note that in the above proof, the fact that both $B_{2,3}$ and $B_{4,5}$ contain at least two vertices is only used to ensure that $A = A_2$. To prove that the colouring is indeed a proper colouring, we do not require $B_{2,3}$ or $B_{4,5}$ to contain at least two vertices. We state this so that some other cases can reduce to this case.

Case 1.2. One of $B_{2,3}$ and $B_{4,5}$ contains at least two vertices while the other contains exactly one vertex.

By symmetry we assume that $B_{2,3}$ contains at least two vertices. By (P20), A_1 and A_4 are empty. By (P24), one of $A_2 \cup A_3$ and A_5 is empty.

Case 1.2.1. A_5 is empty.

By (P19), A_3 is stable. Based on the colouring of Case 1.1, colour $B_{4,5}$ with colour 3 and A_3 with colour 1. By (P2) and (P6), A_3 do not conflict with A_2 or $B_{2,3}$. This gives a proper colouring.

Case 1.2.2. A_5 is nonempty.

By symmetry it is equivalent to the case that $B_{4,5}$ contains at least two vertices while $B_{2,3}$ contains exactly one vertex, and A_2 is nonempty while $A_4 \cup A_5$ is empty. Then it reduces to Case 1.1.

Case 1.3. Both $B_{2,3}$ and $B_{4,5}$ contain exactly one vertex.

Case 1.3.1. One of A_2 and A_5 is nonstable.

By symmetry we assume that A_2 is nonstable. By (P19), A_1 and A_3 are stable while A_4 and A_5 are empty. If A_1 is empty, then it reduces to Case 1.2.1. Now suppose that A_1 is nonempty.

- Colour $v_1, \ldots, v_5$ with colours $2, \omega, 1, 3, 1$ in order.

- Note that $b \in B_{4,5}$ has at most one neighbour in each component of A_2. Colour each component of A_2 with colours in $\{1, \ldots, \omega - 1\}$ so that the neighbour of b is coloured with colour 1, if it exists. Colour A_1 with colour 1 and A_3 with colour 2.

- Colour $B_{2,3}$ with colour 2 and $B_{4,5}$ with colour $\omega - 2$.

- Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $3, \omega - 1, \omega - 2, \omega, \omega - 1$ in order. By (P23),

$C_{2,4}$ and $C_{5,2}$ do not conflict.

Suppose that $a_1 \in A_2$ with colour $\omega - 2$ is adjacent to $c \in C_{3,5}$. Let $b \in B_{4,5}$. Then $a_1 b \notin E$, and there is a vertex $a_2 \in A_2$ adjacent to a_1 such that $a_2 c, a_2 b \notin E$. Then $\{a_2, a_1, c, v_3, v_4, b\}$ induces a P_6. So A and $C_{3,5}$ do not conflict. The reason why $C_{1,3} \cup C_{2,4} \cup C_{5,2}$ does not conflict with A is similar to Case 1.1. So C and A do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $\omega - 1, \omega, \omega - 1, \omega - 1, \omega$ in order. The reason why F does not conflict with A is similar to Case 1.1.

Case 1.3.2. Both A_2 and A_5 are stable.

By (P19), A_1, A_3 and A_4 are stable.

Case 1.3.2.1. A_1 is nonempty.

• Colour $v_1, \ldots, v_5$ with colours $4, 2, 1, 4, 3$ in order.

• Colour $A_1, \ldots, A_5$ with colours $1, 1, 3, 2, 2$ in order.

• Colour $B_{2,3}$ with colour 3 and $B_{4,5}$ with colour 2.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $5, 3, 4, 6, 4$ in order. By (P22), $C_{2,4}$ and A_3 do not conflict. By (P23), $C_{3,5}$ and $C_{5,2}$ do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $6, 6, 6, 6, 5$ in order.

Case 1.3.2.2. A_1 is empty while $A_3 \cup A_4$ is nonempty.

By symmetry we assume that A_3 is nonempty.

• Colour $v_1, \ldots, v_5$ with colours $3, 4, 1, 3, 2$ in order.

• Colour A_2, A_3, A_4, A_5 with colours $2, 2, 1, 1$ in order.

• Colour $B_{2,3}$ with colour 2 and $B_{4,5}$ with colour 1.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $4, 5, 6, 5, 3$ in order. By (P23), $C_{2,4}$ and $C_{4,1}$ do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $6, 6, 6, 5, 6$ in order.

Case 1.3.2.3. $A_1 \cup A_3 \cup A_4$ is empty.

If one of A_2 and A_5 is empty, then it reduces to Case 1.1. Now suppose that both A_2 and A_5 are nonempty.

Now we prove that each vertex in $C_{2,4} \cup C_{3,5}$ is complete to one of A_2 and A_5, and anti-complete to the other. By symmetry, it suffices to show that each vertex in $C_{2,4}$ is complete to one of A_2 and A_5 and anti-complete to the other. Figure 3 is a diagram of the following proof. Let $a_1 \in A_2$, $a_2 \in A_5$, $b \in B_{2,3}$, $c \in C_{2,4}$. By (P3), $a_1 a_2 \in E$. By (P6), $b a_1 \notin E$. By (P12), $bc \notin E$. By (P24), $b a_2 \in E$. If $a_1 c, a_2 c \in E$, then $\{a_1, a_2, c, v_2\}$ induces a diamond. If $a_1 c, a_2 c \notin E$, then $\{c, v_4, v_3, b, a_2, a_1\}$ induces a P_6.

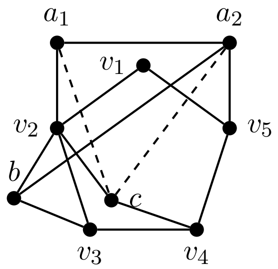

Figure 3: Diagram for Case 1.3.2.3.

- Colour $v_1, \dots, v_5$ with colours $1, 3, 5, 4, 3$ in order.

- Colour A_2 with colour 1 and A_5 with colour 2.

- Colour $B_{2,3}$ with colour 1 and $B_{4,5}$ with colour 2.

- Colour $C_{1,3}, C_{4,1}, C_{5,2}$ with colours $6, 3, 5$ in order. Colour the vertices in $C_{2,4} \cup C_{3,5}$ which is adjacent to A_2 with colour 2, and those adjacent to A_5 with colour 1. Suppose that $c_1 \in C_{2,4}$ and $c_2 \in C_{3,5}$ are adjacent and have the same colour, say 2, then both of them are adjacent to some vertex $a \in A_2$, and so $\{a, c_1, c_2, v_2\}$ induces a diamond. So $C_{2,4}$ and $C_{3,5}$ do not conflict.

- Colour $F_1, \dots, F_5$ with colours $6, 5, 6, 6, 6$ in order.

Case 2. Exactly one of $B_{2,3}$ and $B_{4,5}$ is nonempty.

By symmetry we assume that $B_{2,3}$ is nonempty. By (P19), $\{A_1, \dots, A_5\}$ contains at most two nonstable sets. If there are two nonstable sets, they must be A_2 and A_3; if there is one nonstable set, it can be A_2, A_3 or A_5.

Claim 2. *Suppose that $B_{2,3}$ is nonempty. If $a \in A_5$ and $c \in C_{3,5} \cup C_{5,2}$ are adjacent, then G is ω-colourable.*

Proof. By symmetry suppose that $c \in C_{5,2}$. Let $Q' = \{c, v_2, v_3, v_4, v_5\}$, then Q' induces a C_5. By (P12), c is anti-complete to $B_{2,3}$. Let $b \in B_{2,3}$, then $N_{Q'}(b) = \{v_2, v_3\}$ and $N_{Q'}(a) = \{c, v_5\}$. Then it reduces to Case 1. □

So in Case 2 we assume that A_5 and $C_{3,5} \cup C_{5,2}$ are anti-complete.

Case 2.1. Both A_2 and A_3 are nonstable.

By (P19), A_1, A_4 and A_5 are empty.

- Colour $v_1, \dots, v_5$ with colours $1, \omega, 1, 2, 3$ in order.

- Colour A_3 with colours in $\{2, \dots, \omega\}$. For each component of A_2, colour it with colours in $\{1, \dots, \omega - 1\}$ using the largest colours available.

- Colour $C_{3,5}, C_{4,1}, C_{5,2}$ with colours $\omega, 3, 1$ in order. For each vertex in $C_{1,3}$, colour it with colour 2 if it has a neighbour in $C_{3,5}$, otherwise colour it with colour ω. For each vertex in $C_{2,4}$, colour it with colour $\omega - 1$ if it has a neighbour in $C_{5,2}$, otherwise colour it with colour 1.

Suppose that $a \in A_2$ and $c \in C_{2,4}$ are adjacent and have the same colour 1. Then a is in a component K in A_2 of size $\omega - 1$. Then $K \cup \{v_2, c\}$ is a clique of size $\omega + 1$ by (P9). Suppose that $a \in A_2$ and $c_1 \in C_{2,4}$ are adjacent and have the same colour $\omega - 1$. Then c_1 has a neighbour $c_2 \in C_{5,2}$. By (P25), $ac_2 \notin E$. Then $\{a, v_2, c_1, c_2\}$ induces a diamond. So A_2 and $C_{2,4}$ do not conflict. Similarly, A_3 and $C_{1,3}$ do not conflict.

Now we prove that $C_{4,1}$ is anti-complete to every nontrivial component of A_2 and A_3. By symmetry suppose that $a_1, a_2 \in A_2$ such that $a_1 a_2 \in E$, and a_1 has a neighbour $c \in C_{4,1}$. By (P10), $a_2 c \notin E$, and there is a vertex $a_3 \in A_3$ such that $a_3 c \notin E$ since A_3 is nonstable. Then $\{a_2, a_1, c, v_4, v_3, a_3\}$ induces a P_6. So A and C do not conflict.

- Now we prove that there is at most one vertex in $B_{2,3}$ which has a neighbour in $C_{4,1}$. Since every vertex in $C_{4,1}$ has at most one neighbour in $B_{2.3}$ by (P12), we may assume that $b_1, b_2 \in B_{2,3}$, $c_1, c_2 \in C_{4,1}$ such that $b_1 c_1, b_2 c_2 \in E$. Let a be a vertex in a nontrivial component of A_3. Then $ac_1, ac_2 \notin E$, and so $\{c_2, v_1, c_1, b_1, v_3, a\}$ induces a P_6.

Colour $B_{2,3}$ with colours in $\{2, \dots, \omega - 1\}$ so that the vertex which has a neighbour in $C_{4,1}$ is coloured with colour 2, if it exists. So B and C do not conflict.

- Colour $F_1, \dots, F_5$ with colours $\omega, \omega - 1, \omega, \omega - 1, \omega - 1$ in order.

Case 2.2. Exactly one of A_2 and A_3 is nonstable.

By symmetry we assume that A_2 is nonstable. By (P19), A_4 and A_5 is empty.

• Colour $v_1, \dots, v_5$ with colours $1, \omega, 2, 1, 2$ in order.

• Colour A_2 with colours in $\{1, \dots, \omega - 1\}$, A_1 with colour 2 and A_3 with colour 1.

• Colour $B_{2,3}$ with colours in $\{1, 3, 4, \dots, \omega - 1\}$.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}$ with colours $3, \omega - 1, \omega$ in order. If A_1 is empty, then colour $C_{4,1}$ with colour 2 and $C_{5,2}$ with colour $\omega - 2$, otherwise colour $C_{4,1}$ with colour $\omega - 2$ and $C_{5,2}$ with colour $\omega - 1$.

By (P23), $C_{2,4}$ and $C_{5,2}$ do not conflict. By (P25), A_2 and $C_{5,2}$ do not conflict. If A_1 is nonempty, then $B_{2,3}$ contains only one vertex with colour 1 by (P20). So $B_{2,3}$ and $C_{4,1}$ do not conflict. Suppose that $a_1 \in A_2$ and $c \in C_{4,1}$ are adjacent and have the same colour 2 or $\omega - 2$. Then there is a vertex $a_2 \in A_2$ such that $a_1a_2 \in E$ and $a_2c \notin E$. Let $b \in B_{2,3}$. Then $\{a_2, v_2, b, c, v_4, v_5\}$ or $\{a_2, a_1, c, v_4, v_3, b\}$ induces a P_6, depending on whether b and c are adjacent. So A_2 and $C_{4,1}$ do not conflict.

• Colour $F_1, \dots, F_5$ with colours $\omega, \omega - 1, \omega, \omega - 1, \omega - 1$ in order.

Case 2.3. A_5 is nonstable.

By (P19), A_2 and A_3 are empty.

• Colour $v_1, \dots, v_5$ with colours $\omega - 2, 1, 2, 1, \omega$ in order.

• Colour $B_{2,3}$ with colours in $\{3, \dots, \omega\}$. Exceptionally, when $B_{2,3}$ contains exactly $\omega - 3$ vertices, we use colours $\{3, 4, \dots, \omega - 2, \omega\}$ instead of colours $\{3, \dots, \omega - 1\}$.

• For each component of A_5, sort its vertices by the colours of their neighbours in $B_{2,3}$ in ascending order, putting the vertices with no neighbours in $B_{2,3}$ at the end of the queue. Then colour the vertices in the queue with colours $1, \dots, \omega - 1$ in order. Colour A_1 with colour 1 and A_4 with colour 2.

• Colour $C_{1,3}, C_{2,4}, C_{5,2}$ with colours $3, \omega, \omega - 1$ in order. If A_4 is empty, then colour $C_{3,5}$ with colour $\omega - 2$ and $C_{4,1}$ with colour 2, otherwise colour $C_{3,5}$ with colour 3 and $C_{4,1}$ with colour $\omega - 2$.

By (P23), $C_{1,3}$ and $C_{3,5}$ do not conflict. If A_4 is nonempty, then $B_{2,3}$ contains only one vertex with colour 3 by (P20). So $B_{2,3}$ and $C_{4,1}$ do not conflict. Note that we assume that A_5 and $C_{3,5} \cup C_{5,2}$ are anti-complete. The reason why A_5 and $C_{1,3}$ do not conflict is similar to the reason why A_2 and $C_{4,1}$ do not conflict in Case 1.1. So A_5 and C do not conflict.

• Colour $F_1, \dots, F_5$ with colours $\omega, \omega - 1, \omega - 1, \omega, \omega - 1$ in order.

Case 2.4. None of $A_1, \dots, A_5$ is nonstable.

• Colour $v_1, \dots, v_5$ with colours $1, 3, 2, 1, 3$ in order.

• Colour $A_1, \dots, A_5$ with colours $2, 1, 1, 4, 2$ in order.

• Colour $B_{2,3}$ with colours in $\{1, 4, 5, \dots, \omega\}$. Note that if A_4 is nonempty, then $B_{2,3}$ contains only one vertex with colour 1 by (P20). So A_4 and $B_{2,3}$ do not conflict.

• Colour $C_{2,4}, C_{3,5}, C_{4,1}$ with colours $5, 6, 3$ in order. If A_1 is empty, then colour $C_{5,2}$ with colour 2, otherwise colour $C_{5,2}$ with colour 5. If A_4 is empty, then colour $C_{1,3}$ with colour 4, otherwise colour $C_{1,3}$ with colour 6.

Note that we assume that A_5 and $C_{5,2}$ are anti-complete. By (P23), $C_{1,3}$ and $C_{3,5}$, $C_{2,4}$ and $C_{5,2}$ do not conflict.

- Colour $F_1, \dots, F_5$ with colours $\omega, 5, \omega, 5, \omega$ in order.

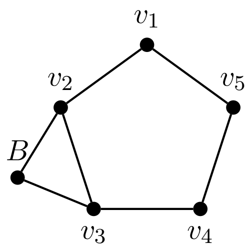

Figure 4: The graph series S_n. B is a clique of size n such that B is complete to $\{v_2, v_3\}$ and anti-complete to $\{v_1, v_4, v_5\}$.

Case 3. B is empty.

Let S_1 be the graph with 6 vertices obtained from C_5 by adding a vertex adjacent to exactly two adjacent vertices of the C_5 (see Figure 4). If G contains an S_1, then it reduces to Case 1 or Case 2. Now assume that G is S_1-free. We have the following two results:

Claim 3. *If G is S_1-free, then $C_{i,i+2}$ is anti-complete to $A_i \cup A_{i+2}$.*

Proof. By symmetry, it suffices to show that $C_{i,i+2}$ is anti-complete to A_i. If $a \in A_i$ and $c \in C_{i,i+2}$ are adjacent, then $\{v_i, c, v_{i+2}, v_{i+3}, v_{i+4}, a\}$ induces an S_1. $\square$

Claim 4. *If G is S_1-free and A_{i+1}, A_{i+3} and A_{i+4} are all nonempty, then each vertex in $C_{i,i+2}$ is complete to one of A_{i+1} and $A_{i+3} \cup A_{i+4}$, and anti-complete to the other.*

Proof. Let $a_1 \in A_{i+1}$, $a_2 \in A_{i+3}$, $a_3 \in A_{i+4}$, $c \in C_{i,i+2}$. By (P2), $a_2a_3 \notin E$. By (P3), $a_1a_2, a_1a_3 \in E$. If $a_1c, a_2c, a_3c \in E$, then $\{a_1, a_2, a_3, c\}$ induces a diamond. If $a_1c, a_2c, a_3c \notin E$, then $\{c, v_i, v_{i+4}, a_3, a_1, a_2\}$ induces a P_6. If $a_1c \notin E$, and exactly one of a_2 and a_3, say a_3, is adjacent to c, then $\{v_i, c, a_3, a_1, a_2, v_{i+3}\}$ induces a P_6. If $a_1c \in E$, and exactly one of a_2 and a_3, say a_2, is adjacent to c, then $\{a_3, a_1, a_2, v_{i+3}, v_{i+4}, c\}$ induces an S_1. $\square$

By (P19), $\{A_1, \dots, A_5\}$ contains at most two nonstable sets, and A_i and A_j are nonstable only if $|i - j| = 1$.

Case 3.1. $\{A_1, \dots, A_5\}$ contains two nonstable sets.

By symmetry, we assume that A_2 and A_3 are nonstable. By (P19), $A_1 \cup A_4 \cup A_5$ is empty.

- Colour $v_1, \dots, v_5$ with colours $1, \omega, 1, 2, 3$, in order.

- Colour A_2 with colours in $\{1, \dots, \omega - 1\}$. By (P16), each component of A_3 contains at most two vertices which have a neighbour in $F_2 \cup F_4$. Colour A_3 with colours in $\{2, \dots, \omega\}$ so that the vertices which have a neighbour in $F_2 \cup F_4$ are coloured with colours in $\{2, 3\}$ if exist.

- Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $2, \omega-1, \omega, 3, 1$ in order. By Claim 3, $C_{2,4} \cup C_{5,2}$ and A_2, $C_{1,3} \cup C_{3,5}$ and A_3 do not conflict. The reason why $C_{4,1}$ and A do not conflict is similar to Case 2.1. So C and A do not conflict.

- Colour $F_1, \dots, F_5$ with colours $\omega, \omega - 1, \omega, \omega - 1, \omega - 1$ in order.

Case 3.2. $\{A_1, \dots, A_5\}$ contains exactly one nonstable set.

By symmetry, assume that A_2 is nonstable. By (P19), $A_4 \cup A_5$ is empty.

- Colour $v_1, \dots, v_5$ with colours $3, \omega, 1, 2, 1$ in order.

- Colour A_2 with colours in $\{1, \dots, \omega - 1\}$, A_1 with colour 1, A_3 with colour 2.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $2, \omega-1, \omega-2, \omega, 3$ in order. By Claim 3, A_2 and $C_{2,4} \cup C_{5,2}$, A_3 and $C_{1,3}$ do not conflict.

Suppose that $a_1 \in A_2$ with colour $\omega - 2$ is adjacent to $c \in C_{3,5}$. Then there is a vertex $a_2 \in A_2$ such that $a_2a_1 \in E$ and $a_2c \notin E$. Then $\{a_1, c, v_5, v_1, v_2, a_2\}$ induces an S_1. So C and A do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $\omega, \omega-1, \omega, \omega-1, \omega-1$ in order.

Case 3.3. $\{A_1, \ldots, A_5\}$ contains no nonstable sets.

Case 3.3.1. One of $\{A_1, \ldots, A_5\}$ is empty.

By symmetry, assume that A_1 is empty.

• Colour $v_1, \ldots, v_5$ with colours $1, 3, 2, 1, 2$ in order.

• Colour A_2, A_3, A_4, A_5 with colours $1, 1, 2, 3$ in order.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $6, 2, 3, 4, 5$ in order. By Claim 3, $C_{2,4}$ and A_4, $C_{3,5}$ and A_5 do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $6, 5, 6, 6, 6$ in order.

Case 3.3.2. None of $\{A_1, \ldots, A_5\}$ are empty.

• Colour $v_1, \ldots, v_5$ with colours $3, 2, 4, 1, 4$ in order.

• Colour $A_1, \ldots, A_5$ with colours $1, 1, 3, 2, 2$ in order.

• Colour $C_{2,4}, C_{4,1}, C_{5,2}$ with colours $4, 5, 6$ in order. By Claim 3, $C_{1,3}$ and A_1, $C_{3,5}$ and A_5 do not conflict. By Claim 4, each vertex in $C_{1,3}$ is complete to one of A_2 and $A_4 \cup A_5$ and anti-complete to the other, and each vertex in $C_{3,5}$ is complete to one of $A_1 \cup A_2$ and A_4 and anti-complete to the other. Colour $C_{1,3}$ and $C_{3,5}$ with colours in $\{1, 2\}$ so that C and A do not conflict. Suppose that $c_1 \in C_{1,3}$, $c_2 \in C_{3,5}$, $c_1c_2 \in E$. If c_1 and c_2 have the same colour, say 1, then both of them are adjacent to some vertex $a \in A_4$, and so $\{a, c_1, c_2, v_3\}$ induces a diamond. So $C_{1,3}$ and $C_{3,5}$ do not conflict.

• Colour $F_1, \ldots, F_5$ with colours $5, 6, 6, 6, 6$ in order.

This completes the proof of Theorem 6. □

5 Computing the Chromatic Number of (P_6, diamond)-Free Graphs in Polynomial Time

In this section we prove the following Theorem 7:

Theorem 7. *The chromatic number of (P_6,diamond)-free graphs can be computed in polynomial time.*

This answers an open question from [7]. To prove Theorem 7, we use Theorem 8 below.

Theorem 8. *The chromatic number of (P_6,diamond,K_6)-free graphs can be computed in polynomial time.*

To prove Theorem 8, we use the following three theorems.

Theorem 9. *There is one 6-vertex-critical (P_6,diamond)-free graph with clique number 3.*

Theorem 10. *There are no 6-vertex-critical (P_6,diamond)-free graphs with clique number 4.*

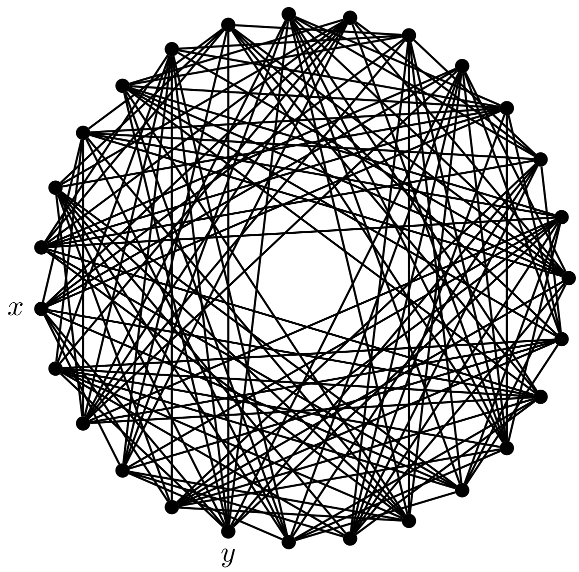

Figure 5: The complement of the 27-vertex Schläfli graph. The unique 6-vertex-critical (P_6,diamond)-free graph $\mathcal{G}$ with clique number 3 is obtained by removing the vertices with labels x and y.

Theorem 11. *There are no 6-vertex-critical (P_6,diamond)-free graphs with clique number 5.*

Proof of Theorem 7. Let G be a (P_6,diamond)-free graph. We can calculate $\vartheta(\overline{G})$ in polynomial time [19]. When $\vartheta(\overline{G}) > 5$, since $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$ [19] and $\chi(G) \leq max\{\omega(G), 6\}$ (i.e. Theorem 5), we have $\chi(G) = \lceil \vartheta(\overline{G}) \rceil$. Then we just have to deal with the case that $\vartheta(\overline{G}) \leq 5$, which implies that $\omega(G) \leq 5$. Then the theorem follows from Theorem 8. □

Note that we could also prove Theorem 7 using the fact that the clique number of a diamond-free graph can be computed in polynomial time: if $\omega(G) \geq 6$, then $\chi(G) = \omega(G)$; otherwise $\omega(G) \leq 5$ and then we use Theorem 8. However, we chose to use the Lovász theta function, since it is a more general strategy that could possibly also be useful for other graph classes.

Proof of Theorem 8. By Theorem 5 and Theorem 3, we only need to consider 5-colouring. By Theorem 2, we need to prove that there are a finite number of 6-vertex-critical (P_6,diamond,K_6)-free graphs. By Theorem 4, every 6-vertex-critical (P_6,diamond)-free graph has clique number at least 3. Then the theorem follows from Theorem 9, Theorem 10, and Theorem 11. □

The unique 6-vertex-critical (P_6,diamond)-free graph $\mathcal{G}$ with clique number 3 from Theorem 9 has 25 vertices and can be obtained from the complement of the Schläfli graph by deleting the vertices labelled x and y in Figure 5. This graph can also be inspected at *the House of Graphs* [12] at: `https://hog.grinvin.org/graphs/45613`.

The proof of Theorem 9 uses computational methods. In the Appendix we give a computer-free proof of a weaker version of Theorem 9 (i.e. Theorem 12). This weaker theorem still suffices to give a complete computer-free proof of Theorem 7 and Theorem 8 (but leads to a less practical algorithm as it does not determine the 6-vertex-critical (P_6,diamond)-free graphs with clique number 3).

Proof of Theorem 9. We used the generation algorithm for k-critical H-free graphs from [8, 18] and extended it to generate 6-vertex-critical (P_6,diamond,K_4)-free graphs. The algorithm terminated in less than 5 minutes and yielded the graph $\mathcal{G}$ as the only 6-vertex-critical (P_6,diamond,K_4)-

free graph. The source code of the program can be downloaded from [17]. We refer to [8, 18] for more details on the algorithm and the proof of its correctness. □

Our polynomial time algorithm for computing the chromatic number of a (P_6,diamond)-free graph G is as follows:

- Step 1. Calculate $\vartheta(\overline{G})$. If $\vartheta(\overline{G}) > 5$, then $\chi(G) = \lceil\vartheta(\overline{G})\rceil$. Otherwise, go to Step 2.
- Step 2. Check whether G contains the unique 6-vertex-critical (P_6,diamond,K_6)-free graph $\mathcal{G}$ from Theorem 9. If yes, then $\chi(G) = 6$. Otherwise, go to Step 3.
- Step 3. Use the algorithm from [11] for 4-colouring of P_6-free graphs to determine whether G is 4-colourable. If not, then $\chi(G) = 5$. Otherwise, go to Step 4.
- Step 4. Use the algorithm from [11] for 4-colouring of P_6-free graphs to determine whether $G \vee K_1$ is 4-colourable. If not, then $\chi(G) = 4$. Otherwise, go to Step 5.
- Step 5. Check whether G contains a C_5. If yes, then $\chi(G) = 3$. Otherwise, $\chi(G) = \vartheta(\overline{G})$.

The remainder of this section is organized as follows. In the first subsection we prove some lemmas which we will use in the next two subsections to prove Theorem 10 and Theorem 11, and in the Appendix to prove Theorem 12.

5.1 Lemmas

Lemma 2. *Every 6-vertex-critical (P_6,diamond)-free graph with clique number ω ($\omega = 4, 5$) is $S_{\omega-2}$-free. (See Figure 4 for S_n.)*

Proof. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number ω containing an $S_{\omega-2}$. In the proof of this lemma, we follow the partitioning of V from Section 3. Let the C_5 in the $S_{\omega-2}$ be Q, and the other two or three vertices in the $S_{\omega-2}$, denoted by b_1, b_2 (and b_3 if it exists), be in $B_{2,3}$. By $\omega(G) = \omega$, $B_{2,3}$ contains no vertices other than these two or three vertices, and F_5 is empty. By (P20), $A_1 \cup A_4$ is empty. By (P24), one of $A_2 \cup A_3$ and A_5 is empty. Assume that $B = B_{2,3} \cup B_{4,5}$. Then the edges between $B_{2,3}$ and $B_{4,5}$ form a matching.

Claim 5. $B = B_{2,3}$.

Proof. Suppose that $B_{4,5}$ is nonempty.

Now we prove that if $u_1 \in B_{2,3}$ and $u_2 \in B_{4,5}$ are nonadjacent, then both u_1 and u_2 have no neighbours in C. Assume that $c \in C_{4,1}$ is adjacent to u_1. Then $\{v_3, u_1, c, v_1, v_5, u_2\}$ induces a P_6. So u_1 and $C_{4,1}$ are anti-complete. Similarly, u_2 and $C_{1,3}$ are anti-complete. By (P12), both u_1 and u_2 have no neighbours in C.

We discuss two cases based on the value of ω, and give a contradiction for each case.

Case 1. ω=4.

Let $b_4 \in B_{4,5}$. Then one of b_1 and b_2, say b_1, is nonadjacent to b_4. Then both b_1 and b_4 have no neighbours in C.

If A_5 is empty, we consider the degree of b_1. Since b_1 has two neighbours v_2, v_3 in Q, one neighbour b_2 in $B_{2,3}$, at most one neighbour in $B_{4,5}$, and no neighbours in $A \cup C \cup F$, we have $d(b_1) \leq 4$.

If $A_2 \cup A_3$ is empty, we consider the degree of b_4. Since b_4 has two neighbours v_4, v_5 in Q, at most one neighbour in $B_{2,3}$, at most one neighbour in $B_{4,5} \cup F_2$, and no neighbours in $A \cup C$, we have $d(b_4) \le 4$.

Case 2. ω=5.

Now we prove that either $B_{4,5}$ contains at most one vertex, or $B_{2,3}$ and $B_{4,5}$ are anti-complete. Suppose that $b_4, b_5 \in B_{4,5}$ such that $b_4 b_1 \in E$. We may assume that $b_5 b_2 \notin E$. Recall that if $u_1 \in B_{2,3}$ and $u_2 \in B_{4,5}$ are nonadjacent, then both u_1 and u_2 have no neighbours in C. Since both $B_{2,3}$ and $B_{4,5}$ contain at least two vertices, $B_{2,3}$ and $C_{4,1}$, $B_{4,5}$ and $C_{1,3}$ are anti-complete. If $c \in C_{4,1}$, then $\{b_2, b_1, b_4, v_5, v_1, c\}$ induces a P_6. So $C_{4,1}$ is empty. If $f \in F_3$, then $\{v_1, f, v_3, b_1, b_4, b_5\}$ induces a P_6. So F_3 is empty. Similarly, $C_{1,3} \cup F_4$ is empty. Then $d(v_1) \le 3$. So either $B_{4,5}$ contains at most one vertex, or $B_{2,3}$ and $B_{4,5}$ are anti-complete.

Let $b_4 \in B_{4,5}$. Suppose that b_4 has no neighbours in $A_2 \cup A_3$. If $B_{4,5}$ contains only one vertex, since b_4 has two neighbours v_4, v_5 in Q, at most one neighbour in $B_{2,3}$, at most one neighbour in F_2, and no neighbours in $A \cup C$, we have $d(b_4) \le 4$. If $B_{2,3}$ and $B_{4,5}$ are anti-complete, since b_4 has two neighbours v_4, v_5 in Q, at most two neighbours in $B_{4,5} \cup F_2$, and no neighbours in $B_{2,3} \cup A \cup C$, we have $d(b_4) \le 4$. So b_4 has a neighbour in $A_2 \cup A_3$. Then A_5 is empty. By symmetry we may assume that $b_1 b_4 \notin E$, then b_1 has no neighbour in $C_{4,1}$. Since b_1 has two neighbours v_2, v_3 in Q, two neighbours b_2, b_3 in $B_{2,3}$, no neighbours in $A \cup B_{4,5} \cup C$, we have $d(b_1) = 4$. $\square$

Then by $\delta(G) \ge 5$, each vertex of $B_{2,3}$ has at least $6 - \omega$ neighbours in $C_{4,1} \cup A_5$. Moreover, no pairs of vertices in $B_{2,3}$ can have a common neighbour in $C_{4,1} \cup A_5$ since G is diamond-free.

Claim 6. *$A_2 \cup A_3$ is empty.*

Proof. By symmetry suppose that $a \in A_2$. Then A_5 is empty and b_1 has a neighbour $c \in C_{4,1}$. Then $\{b_2, v_2, a, c, v_4, v_5\}$ or $\{a, v_2, b_1, c, v_4, v_5\}$ induces a P_6, depending on whether a and c are adjacent. $\square$

Claim 7. *$C_{5,2} \cup C_{3,5}$ and A_5 are anti-complete.*

Proof. By symmetry, it suffices to show that $C_{5,2}$ and A_5 are anti-complete. Suppose that $c \in C_{5,2}$ and $a \in A_5$ are adjacent. Let $Q' = \{c, v_2, v_3, v_4, v_5\}$. Then $N_{Q'}(b_1) = N_{Q'}(b_2) = \{v_2, v_3\}$, $N_{Q'}(a) = \{c, v_5\}$, moreover $N_{Q'}(b_3) = \{v_2, v_3\}$ if b_3 exists. This contradicts with Claim 5. $\square$

Claim 8. *If $a \in A_5$ has a neighbour in $B_{2,3}$, then a has no neighbours in A_5.*

Proof. Suppose that $a_1, a_2 \in A_5$ such that $a_1 b_1, a_1 a_2 \in E$. Then $a_2 b_1, a_1 b_2 \notin E$, moreover $a_1 b_3 \notin E$ if b_3 exists. Let $Q' = \{a_1, b_1, v_2, v_1, v_5\}$. Then $N_{Q'}(b_2) = N_{Q'}(v_3) = \{b_1, v_2\}$, $N_{Q'}(a_2) = \{a_1, v_5\}$, moreover $N_{Q'}(b_3) = \{b_1, v_2\}$ if b_3 exists. This contradicts with Claim 5. $\square$

Claim 9. *If $a \in A_5$ has a neighbour $b \in B_{2,3}$, then a is complete to $C_{4,1} \setminus N_{C_{4,1}}(b)$.*

Proof. Suppose that $c \in C_{4,1}$ such that $ac, bc \notin E$, then $\{c, v_4, v_5, a, b, v_2\}$ induces a P_6. $\square$

Claim 10. *If $c \in C_{4,1}$ has a neighbour $b \in B_{2,3}$, then c is complete to $A_5 \setminus N_{A_5}(b)$, and A_5 is stable.*

Proof. Suppose that $a \in A_5$ such that $ac, ab \notin E$, then $\{v_3, b, c, v_1, v_5, a\}$ induces a P_6. So c is complete to $A_5 \setminus N_{A_5}(b)$. Suppose that $a_1, a_2 \in A_5$ such that $a_1 a_2 \in E$. Then $a_1 b, a_2 b \notin E$ by Claim 8, and so $a_1 c, a_2 c \in E$. Now $\{a_1, a_2, c, v_5\}$ induces a diamond. $\square$

Claim 11. *$C_{2,4} \cup C_{1,3}$ and A_5 are complete.*

Proof. By symmetry, it suffices to show that $C_{2,4}$ and A_5 are complete. Suppose that $c \in C_{2,4}$ and $a \in A_5$ are nonadjacent. Since a has at most one neighbour in $B_{2,3}$, we may assume that $b_1 a \notin E$. Then $\{b_1, v_2, c, v_4, v_5, a\}$ induces a P_6. □

Claim 12. *If there are at least two vertices in $B_{2,3}$ which have a neighbour in $C_{4,1}$, then $C_{4,1}$ and $C_{5,2} \cup C_{3,5}$ are complete.*

Proof. By symmetry, it suffices to show that $C_{4,1}$ and $C_{5,2}$ are complete. Figure 6 is a diagram of the following proof. Let $c_1, c_2 \in C_{4,1}$ such that $c_1 b_1, c_2 b_2 \in E$, $c_3 \in C_{5,2}$. Suppose that $c_3 c_1, c_3 c_2 \notin E$, then $\{c_1, v_4, c_2, b_2, v_2, c_3\}$ induces a P_6. So c_3 is adjacent to at least one of c_1 and c_2. By symmetry assume that $c_1 c_3 \in E$. Let c_4 be an arbitrary vertex in $C_{4,1} \setminus \{c_1\}$ with $c_4 c_3 \notin E$. Then $\{c_4, b_2, b_1, c_1, c_3, v_5\}$ or $\{c_4, v_4, c_1, c_3, v_2, b_2\}$ induces a P_6, depending on whether c_4 and b_2 are adjacent. (Note that c_4 is adjacent to at most one of b_1 and b_2.) □

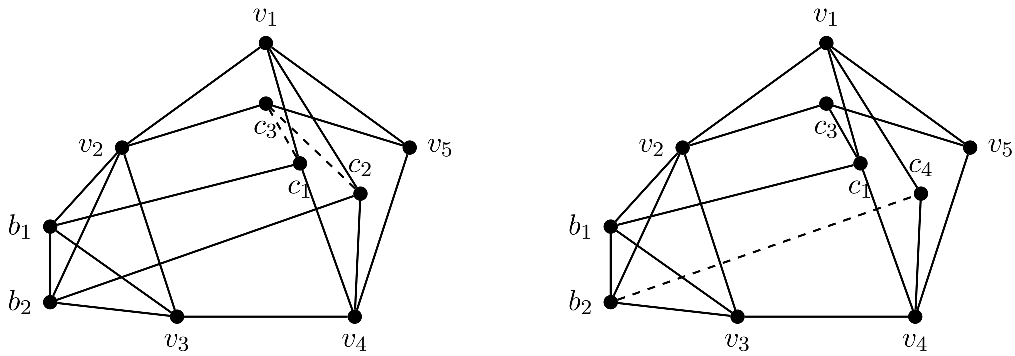

Figure 6: Diagram for Claim 12.

Claim 13. *If there are at least two vertices in $B_{2,3}$ which have a neighbour in $C_{4,1}$, then $C_{4,1}$ and $C_{2,4} \cup C_{1,3}$ are anti-complete.*

Proof. Let both b_1 and b_2 have a neighbour in $C_{4,1}$. By symmetry, it suffices to show that $C_{4,1}$ and $C_{2,4}$ are anti-complete. Let $c_1 \in C_{2,4}$. Since G is diamond-free, c_1 has at most one neighbour in $C_{4,1}$. Suppose that $c_2 \in C_{4,1}$ is adjacent to c_1. Since c_2 is adjacent to at most one vertex in $\{b_1, b_2\}$, assume that $b_1 c_2 \notin E$. Let $c_3 \in C_{4,1}$ be a neighbour of b_1. Then $\{c_1, c_2, v_1, c_3, b_1, v_3\}$ induces a P_6. □

Now we give a 5-colouring of G as follows.

Case 1. There are at least two vertices in $B_{2,3}$ which have a neighbour in $C_{4,1}$.

By Claim 10, A_5 is stable.

- Colour $v_1, \ldots, v_5$ with colours $4, 3, 4, 2, 1$ in order.
- Colour b_1, b_2, b_3 with colours 1,2,5 in order.
- Colour A_5 with colour 4.
- Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $3, 1, 2, 3, 2$ in order. By Claim 13, $C_{1,3}$ and $C_{4,1}$ do not conflict. Note that $C_{4,1}$ contains a $2K_1$. So $C_{5,2}$ and $C_{3,5}$ are anti-complete by Claim 12 since G is diamond-free.
- Colour F with colour 5. Note that F_5 is empty, so F and $B_{2,3}$ do not conflict by (P17).

• Let K be an arbitrary component of Z. If K contains only one vertex, colour it with colour 4. If K contains at least two vertices, then either K is 3-colourable if K has no neighbours in C, or $K \cup N_C(K)$ is a clique of size at most 4 if K has a neighbour in C (cf. the proof of Claim 1). So K can be coloured with colours in $\{1, 2, 3, 4\}$. Colour each component of Z this way.

Case 2. There is at most one vertex in $B_{2,3}$ which has a neighbour in $C_{4,1}$.

Let b_1 have no neighbours in $C_{4,1}$. Moreover if $\omega = 5$, let b_2 have no neighbours in $C_{4,1}$.

• Colour $v_1, \ldots, v_5$ with colours $4, 3, 4, 3, 1$ in order.

• Colour b_1, b_2, b_3 with colours $1, 2, 5$ in order.

• Colour each nontrivial component of A_5 with colours in $\{2, 3, 4, 5\}$. Colour each trivial component of A_5 with colour 3. By Claim 8, A_5 and $B_{2,3}$ do not conflict.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $1, 1, 3, 1, 4$ in order. By Claim 7, $C_{5,2} \cup C_{3,5}$ and A_5 do not conflict.

If $\omega = 4$, then b_1 has at least two neighbours in A_5. If $\omega = 5$, then both b_1 and b_2 have at least one neighbour in A_5. So there are two nonadjacent vertices $a_1, a_2 \in A_5$ which are complete to $C_{4,1}$ by Claim 8 and Claim 9. Note that $\{a_1, a_2\}$ and $C_{2,4} \cup C_{1,3}$ are complete by Claim 11. Then $C_{4,1}$, $C_{2,4}$ and $C_{1,3}$ are pairwise anti-complete.

• Colour F with colour 5. By (P14), $F_2 \cup F_3$ is anti-complete to A_5. If $a_1 \in A_5$ with colour 5 and $f \in F_1 \cup F_4$ are adjacent, by symmetry assume that $f \in F_1$. Let $a_2 \in A_5$ be adjacent to a_1, then $a_2 f \notin E$ by (P16), and so $\{a_2, a_1, f, v_1, v_2, b_1\}$ induces a P_6. So A_5 and F do not conflict.

• The colouring of Z is similar to Case 1.

So G is 5-colourable, a contradiction. □

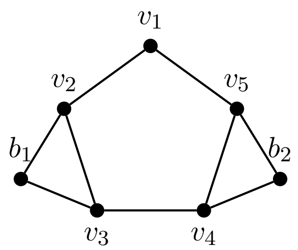

Figure 7: The graph D_1.

Lemma 3. *Every 6-vertex-critical (P_6,diamond)-free graph with clique number 3 is D_1-free. (See Figure 7 for D_1.)*

Proof. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 3 containing a D_1. In the proof of this lemma, we follow the partitioning of V from Section 3. Let the C_5 in the D_1 be Q, and the other two vertices $b_1 \in B_{2,3}$, $b_2 \in B_{4,5}$ such that $b_1 b_2 \notin E$. By $\omega(G) = 3$, $B = \{b_1, b_2\}$.

Claim 14. $C_{1,3} \cup C_{4,1} \cup A_3 \cup A_4 \cup F_2 \cup F_5$ *is empty.*

Proof. By symmetry, it suffices to show that $C_{4,1} \cup A_4 \cup F_2$ is empty. Suppose that $c \in C_{4,1}$, then $\{v_3, b_1, c, v_1, v_5, b_2\}$ or $\{b_1, v_2, v_1, c, v_4, b_2\}$ induces a P_6, depending on whether b_1 and c are adjacent. Suppose that $a \in A_4$, then $\{v_1, v_2, b_1, a, v_4, b_2\}$ induces a P_6. Suppose that $f \in F_2$, then $\{b_2, v_4, v_5, f\}$ induces a K_4 or a diamond. □

Claim 15. *A_1 contains at most one vertex.*

Proof. Suppose that $a_1, a_2 \in A_1$. By the properties from Section 3, A_1 is stable, complete to $B_{2,3} \cup B_{4,5} \cup C_{5,2} \cup C_{2,4} \cup C_{3,5}$ and anti-complete to $A_2 \cup A_5 \cup F_3 \cup F_4 \cup Z$.

Suppose that $f \in F_1$ such that $a_1 f \in E$, then $a_2 f \notin E$ as the graph is diamond-free, then $\{a_2, b_1, a_1, f, v_4, v_5\}$ induces a P_6. So A_1 is anti-complete to F_1. Then a_1 and a_2 are comparable. □

Then by $\delta(G) \geq 5$, b_1 has at least two neighbours in A_5, and b_2 has at least two neighbours in A_2. Let $a_1, a_2 \in N_{A_2}(b_2)$. By the properties from Section 3, A_2 and A_5 are stable, and A_2 is complete to $A_5 \cup B_{4,5}$ and anti-complete to $A_1 \cup B_{2,3} \cup C_{5,2} \cup F_4 \cup Z$. Now we prove that a_1 and a_2 are comparable by showing that their neighbourhoods in $C_{2,4} \cup C_{3,5} \cup F_1 \cup F_3$ are equal.

- $\{a_1, a_2\}$ is anti-complete to $C_{2,4}$: Suppose that $c \in C_{2,4}$ such that $a_1 c \in E$, then c has no neighbours in $A_2 \cup A_5$ other than a_1 because of the diamond-freeness. Let $a_3 \in A_5$, then $\{a_2, a_3, a_1, c, v_4, v_3\}$ induces a P_6.

- For each vertex $c \in C_{3,5}$, either $a_1 c, a_2 c \in E$ or $a_1 c, a_2 c \notin E$: Suppose that $a_1 c \in E$, $a_2 c \notin E$. Let $a_3 \in A_5$, then $a_3 c \notin E$, then $\{a_2, a_3, a_1, c, v_3, v_4\}$ induces a P_6.

- $\{a_1, a_2\}$ is complete to F_1: Suppose that $f \in F_1$ such that $a_1 f \notin E$, then $\{a_1, b_2, v_5, v_1, f, v_3\}$ induces a P_6.

- $\{a_1, a_2\}$ is complete to F_3: Suppose that $f \in F_3$ such that $a_1 f \notin E$, then $\{a_1, b_2, v_5, f, v_3, b_1\}$ induces a P_6.

Then a_1 and a_2 are comparable. This completes the proof of Lemma 3. □

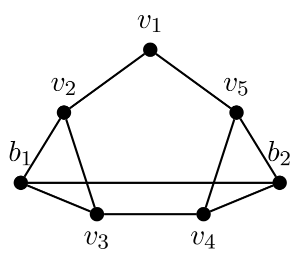

Figure 8: The graph D_2.

Lemma 4. *Every 6-vertex-critical (P_6,diamond)-free graph with clique number 3 is D_2-free. (See Figure 8 for D_2.)*

Proof. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 3 containing a D_2. In the proof of this lemma, we follow the partitioning of V in Section 3. Let the C_5 in the D_2 be Q, and the other two vertices $b_1 \in B_{2,3}$, $b_2 \in B_{4,5}$ such that $b_1 b_2 \in E$. Then $B = \{b_1, b_2\}$. By the properties from Section 3 and $\omega(G) = 3$, A_1, A_3 and A_4 are stable, and $F_2 \cup F_5$ is empty.

Claim 16. *One of A_1 and $A_3 \cup A_4$ is empty.*

Proof. Suppose that $a_1 \in A_1$, $a_2 \in A_3$, then $\{a_1, a_2, b_1, b_2\}$ induces a diamond. □

Claim 17. *$B_{2,3}$ and $C_{4,1}$, $B_{4,5}$ and $C_{1,3}$ are complete.*

Proof. Suppose that $b \in B_{2,3}$ and $c \in C_{4,1}$ are nonadjacent, then $\{v_3, b_1, b_2, v_5, v_1, c\}$ induces a P_6. □

Claim 18. *A_2 and $C_{4,1}$, A_5 and $C_{1,3}$ are complete.*

Proof. Suppose that $a \in A_2$ and $c \in C_{4,1}$ are nonadjacent, then $\{a, v_2, b_1, c, v_4, v_5\}$ induces a P_6. □

Claim 19. *If A_1 is nonempty, then A_2 and $B_{4,5}$, A_5 and $B_{2,3}$ are complete.*

Proof. By symmetry let $a_1 \in A_2$ such that $a_1b_2 \notin E$. If $a_2 \in A_1$, then $\{a_1, v_2, v_1, a_2, b_2, v_4\}$ induces a P_6. □

Claim 20. *A_2 and A_5 are stable.*

Proof. By symmetry suppose that A_2 is nonstable. Then $A_4 \cup A_5$ is empty. If $a \in A_1$, then $A_2 \cup \{v_2, b_2\}$ contains a diamond by Claim 19. If $c \in C_{4,1}$, then $A_2 \cup \{v_2, c\}$ contains a diamond by Claim 18. So $A_1 \cup C_{4,1}$ is empty. Since b_1 has two neighbours v_2, v_3 in Q, a neighbour b_2, and no neighbours in $A \cup C \cup F$, $d(b_1) \leq 3$. □

Claim 21. *A_3 and $C_{1,3}$, A_4 and $C_{4,1}$ are anti-complete.*

Proof. By symmetry suppose that $a \in A_3$, $c \in C_{1,3}$ such that $ac \in E$. By Claim 17, $cb_2 \in E$. Then $\{a, c, b_2, v_3\}$ induces a diamond. □

Now we give a 5-colouring of G as follows.

Suppose that $V \setminus Z$ has a 5-colouring such that one colour of $1, \ldots, 5$ is not used in $C_{1,3} \cup C_{4,1} \cup F$, and F is coloured with at most 2 colours, then we can extend it to a 5-colouring of G as follows. For each component K of Z, if K is trivial, colour it with the colour not used in $C_{1,3} \cup C_{4,1} \cup F$, since Z and $C_{2,4} \cup C_{3,5} \cup C_{5,2}$ are anti-complete by (P26). If K is nontrivial, then either $K \cup N_C(K)$ is a clique of size at most 3 if K has a neighbour in C, or K is 3-colourable if K has no neighbours in C, then K can be coloured with the three colours not used in F. Therefore, it remains to give a colouring of $V \setminus Z$, so that one colour of $1, \ldots, 5$ is not used in $C_{1,3} \cup C_{4,1} \cup F$, and F is coloured with at most 2 colours.

Case 1. $A_3 \cup A_4$ is empty.

Case 1.1. One of A_2 and A_5 is empty.

By symmetry assume that A_5 is empty.

- Colour $v_1, \ldots, v_5$ with colours $2, 3, 1, 3, 1$ in order.
- Colour A_1 and A_2 with colour 1.
- Colour $B_{2,3}$ with colour 5, $B_{4,5}$ with colour 2.
- Colour $C_{1,3}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $3, 5, 4, 2$ in order. For each vertex in $C_{2,4}$, colour it with colour 1 if it has a neighbour in $C_{5,2}$, otherwise colour it with colour 2. If there is a vertex in $C_{2,4}$ with colour 1, then A_1 is empty by (P23). If $a \in A_2$ and $c_1 \in C_{2,4}$ with colour 1 are adjacent, then there is a vertex $c_2 \in C_{5,2}$ adjacent to c_1, and so $\{a, v_2, c_1, c_2\}$ induces a diamond or K_4. So C and A do not conflict.
- Colour F_1, F_3, F_4 with colours $5, 5, 4$ in order.

Case 1.2. Both A_2 and A_5 are nonempty.

Each vertex in $C_{2,4} \cup C_{3,5}$ is complete to one of A_2 and A_5, and anti-complete to the other. The proof can be reviewed in Case 1.3.2.3 of Theorem 6, Section 4.

- Colour $v_1, \ldots, v_5$ with colours $2, 3, 4, 3, 5$ in order.

- Colour A_1, A_2, A_5 with colours $4, 1, 2$ in order.
- Colour $B_{2,3}$ with colour 1, $B_{4,5}$ with colour 2.
- Colour $C_{1,3}$ with colour 3, $C_{4,1}$ with colour 5. Colour $C_{5,2}$ with colour 2 if A_1 is nonempty, otherwise colour $C_{5,2}$ with colour 4. Colour the vertices in $C_{2,4} \cup C_{3,5}$ which are adjacent to A_2 with colour 2, and those adjacent to A_5 with colour 1. Note that $C_{5,2}$ is anti-complete to A_5 by (P25), and anti-complete to $C_{2,4} \cup C_{3,5}$ when A_1 is nonempty by (P23).
- Colour F_1, F_3, F_4 with colours $5, 3, 5$ in order.

Case 2. $A_3 \cup A_4$ is nonempty.

Then A_1 is empty by Claim 16. By symmetry assume that A_3 is nonempty.

- Colour $v_1, \ldots, v_5$ with colours $1, 4, 3, 1, 5$ in order.
- Colour A_2, A_3, A_4, A_5 with colours $1, 1, 2, 3$ in order.
- Colour $B_{2,3}$ with colour 1, $B_{4,5}$ with colour 2.
- Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $5, 2, 4, 2, 3$ in order. Note that $C_{2,4}$ and $C_{4,1}$ are anti-complete by (P23), $C_{2,4}$ and A_4 are anti-complete by (P25), $C_{4,1}$ and A_4 are anti-complete by Claim 21.
- Colour F_1, F_3, F_4 with colours $5, 4, 5$ in order.

So G is 5-colourable, a contradiction. □

Lemma 5. *Every 6-vertex-critical (P_6,diamond)-free graph with clique number 3 is S_1-free. (See Figure 4 for S_1.)*

Proof. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 3 containing an S_1. In the proof of this lemma, we follow the partitioning of V from Section 3. Let the C_5 in the S_1 be Q, and the other vertex b in $B_{2,3}$. By Lemma 3 and Lemma 4, G is (D_1,D_2)-free, and so $B = \{b\}$. Since $\omega(G) = 3$, F_5 is empty.

Claim 22. *$C_{5,2} \cup C_{3,5}$ and A_5 are anti-complete.*

Proof. By symmetry suppose that $c \in C_{5,2}$ and $a \in A_5$ are adjacent. Then $\{c, v_2, v_3, v_4, v_5, b, a\}$ induces a D_1 or D_2. □

Claim 23. *$A_1 \cup A_4$ and $C_{4,1}$ are anti-complete.*

Proof. By symmetry suppose that $a \in A_1$ and $c \in C_{4,1}$ are adjacent. By (P7), $ab \in E$. Then either $\{a, b, c, v_1\}$ induces a diamond, or $\{v_1, v_2, v_3, v_4, c, b, a\}$ induces a D_2, depending on whether c and b are adjacent. □

Now we give a 5-colouring of G as follows.

Suppose that one colour of $1, \ldots, 5$ is not used in $C_{1,3} \cup C_{2,4} \cup C_{4,1} \cup F$, and F is coloured with at most 2 colours. For each component K of Z, if K is trivial, colour it with the colour not used in $C_{1,3} \cup C_{2,4} \cup C_{4,1} \cup F$, since Z and $C_{3,5} \cup C_{5,2}$ are anti-complete by (P26). If K is nontrivial, then either $K \cup N_C(K)$ is a clique of size at most 3 if K has a neighbour in C, or K is 3-colourable if K has no neighbours in C, then K can be coloured with the three colours not used in F. Now we give a colouring of $V \setminus Z$, so that one colour of $1, \ldots, 5$ is not used in $C_{1,3} \cup C_{2,4} \cup C_{4,1} \cup F$, and F is coloured with at most 2 colours. We consider four cases which are similar to the subcases of Case 2 in Section 4.

Case 1. Both A_2 and A_3 are nonstable.

• Colour $v_1, \dots, v_5$ with colours $1, 3, 1, 3, 2$ in order.

• Colour A_2 with colours in $\{1, 2\}$ so that the trivial components are coloured with colour 1. Colour A_3 with colours in $\{2, 3\}$ so that the trivial components are coloured with colour 2.

• Colour b with colour 2.

• Colour $C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $4, 3, 5, 1$ in order. For each vertex in $C_{1,3}$, colour it with colour 2 if it has a neighbour in $C_{3,5}$, otherwise colour it with colour 3.

• Colour F_1, F_2, F_3, F_4 with colours $5, 5, 5, 5$ in order.

Case 2. Exactly one of A_2 and A_3 is nonstable.

By symmetry assume that A_2 is nonstable. If A_1 is empty, then it reduces to Case 1. Now suppose that A_1 is nonempty.

• Colour $v_1, \dots, v_5$ with colours $1, 4, 2, 1, 2$ in order.

• Colour A_2 with colours in $\{1, 3\}$ so that the trivial components are coloured with colour 1. Colour A_1 with colour 2, A_3 with colour 1.

• Colour b with colour 1.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}, C_{5,2}$ with colours $4, 3, 5, 2, 3$ in order. By Claim 23, A_1 and $C_{4,1}$ do not conflict.

• Colour F_1, F_2, F_3, F_4 with colours $5, 5, 5, 3$ in order.

Case 3. A_5 is nonstable.

Now we prove that $C_{1,3} \cup C_{2,4}$ is empty. By symmetry suppose that $c \in C_{1,3}$. Since A_5 is nonstable, let $a_1, a_2 \in A_5$ such that $a_1a_2 \in E$. Then at least one of a_1 and a_2 is nonadjacent to b_1, assume that $a_1b \notin E$. Then $ca_1 \in E$, or else $\{b, v_3, c, v_1, v_5, a_1\}$ induces a P_6. So $a_2b \in E$, or else $ca_2 \in E$ and then $\{c, a_1, c_2, v_5\}$ induces a diamond. Then $\{b, v_2, v_1, v_5, a_2, a_1, v_3\}$ induces a D_1. So $C_{1,3} \cup C_{2,4}$ is empty.

• Colour $v_1, \dots, v_5$ with colours $4, 2, 4, 1, 5$ in order.

• Colour A_5 with colours in $\{1, 2\}$ so that the trivial components are coloured with colour 1. Colour A_1 with colour 1, A_4 with colour 2.

• Colour b with colour 3.

• Colour $C_{3,5}, C_{4,1}, C_{5,2}$ with colours $3, 5, 4$ in order.

• Colour F_1, F_2, F_3, F_4 with colours $5, 3, 3, 5$ in order.

Case 4. None of $A_1, \dots, A_5$ is nonstable.

Case 4.1. $A_1 \cup A_4$ is nonempty.

By symmetry assume that A_4 is nonempty.

• Colour $v_1, \dots, v_5$ with colours $2, 3, 1, 2, 3$ in order.

• Colour $A_1, \dots, A_5$ with colours $1, 2, 2, 3, 1$ in order.

• Colour b with colour 4.

• Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{4,1}$ with colours $5, 4, 5, 3$ in order. If A_1 is empty, then colour $C_{5,2}$ with colour 1, otherwise colour $C_{5,2}$ with colour 4. By (P23), $C_{2,4}$ and $C_{5,2}$, $C_{1,3}$ and $C_{3,5}$ are anti-complete. By Claim 22, $C_{5,2}$ and A_5 do not conflict. By Claim 23, A_4 and $C_{4,1}$ do not conflict.

• Colour F_1, F_2, F_3, F_4 with colours $5, 4, 5, 4$ in order.

Case 4.2. $A_1 \cup A_4$ is empty.

- Colour $v_1, \ldots, v_5$ with colours $1, 2, 1, 2, 4$ in order.
- Colour A_2, A_3, A_5 with colours $1, 2, 3$ in order.
- Colour b with colour 5.
- Colour $C_{1,3}, C_{2,4}, C_{3,5}, C_{5,2}$ with colours $4, 5, 2, 1$ in order. For each vertex in $C_{4,1}$, colour it with colour 3 if it has a neighbour in $C_{1,3}$, otherwise colour it with colour 4. Suppose that $c_1 \in C_{4,1}$ with colour 3 is adjacent to $a \in A_5$, then there is a vertex $c_2 \in C_{1,3}$ adjacent to c_1, then either $\{v_1, c_1, c_2, a\}$ induces a diamond, or $\{v_2, v_3, c_2, c_1, a, v_5\}$ induces a P_6, depending on whether a and c_2 are adjacent. So $C_{4,1}$ and A_5 do not conflict.
- Colour F_1, F_2, F_3, F_4 with colours $5, 5, 3, 5$ in order.

So G is 5-colourable, a contradiction. □

Lemma 6. *Every 6-vertex-critical (P_6,diamond)-free graph with clique number ω ($\omega = 3, 4, 5$) is ($K_\omega + K_1$)-free.*

Proof. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number ω. By Lemma 2 and Lemma 5, G is $S_{\omega-2}$-free. Let $A = \{a_1, \ldots, a_\omega\}$ be a K_ω in G. Let $B_i = \{v \in V \setminus A : va_i \in E\}$ for $i = 1, \ldots, \omega$, $B = \bigcup_{i=1}^{\omega} B_i$, $C = V \setminus (A \cup B)$. Since A is an arbitrary K_ω, we just need to prove that C is empty.

Since G is (diamond,$K_{\omega+1}$)-free and $\delta(G) \geq 5$, we have some obvious properties of B:

- $B_1, \ldots, B_\omega$ are pairwise disjoint.
- Each component of B_i is a clique of size at most $\omega - 1$.
- Every $b \in B_i$ has at most one neighbour in each component of B_j for $i \neq j$.
- Each B_i contains at least $6 - \omega$ vertices.

Claim 24. *Each vertex in C is anti-complete to B_i for some $i \in \{1, \ldots, \omega\}$.*

Proof. Let c be an arbitrary vertex in C. Since every B_i is nonempty, suppose that $\{b_1, \ldots, b_\omega\} \subseteq N_B(c)$ such that $b_j \in B_j$ for $j = 1, \ldots, \omega$. Since G is $S_{\omega-2}$-free, $b_1, \ldots, b_\omega$ are pairwise adjacent. Then $\{b_1, \ldots, b_\omega, c\}$ induces a $K_{\omega+1}$. □

Claim 25. *No vertices in C have a neighbour in B_i and a neighbour B_j for $i \neq j$.*

Proof. Suppose that $c_1 \in C$ has two neighbours $b_1 \in B_1$ and $b_2 \in B_2$. Since G is $S_{\omega-2}$-free, $b_1 b_2 \in E$. Moreover, for each neighbour b of c_1 in B, $bb_1 \in E$ if $b \notin B_1$, and $bb_2 \in E$ if $b \notin B_2$. So $N_B(c_1)$ is connected, and forms a clique since G is diamond-free. Since G has no clique cutsets, $G - N_B(c_1)$ is connected. Assume that $c_1, c_2, \ldots, c_n, b_3, a_i$ ($n \geq 2$) is a shortest path between c_1 and A in $G - N_B(c_1)$. Then $c_2, \ldots, c_n \in C$. If $n \geq 3$, then $\{c_1, c_2, \ldots, c_n, b_3, a_i, a_j\}$ contains a P_6 for some j. So $n = 2$.

Since $\{c_1, c_2, b_1, b_2\}$ cannot induce a diamond, either $b_1c_2, b_2c_2 \in E$ or $b_1c_2, b_2c_2 \notin E$. Assume that $b_1c_2, b_2c_2 \in E$. By symmetry assume that $b_3 \notin B_1$. Then $b_1b_3 \in E$, and so $\{c_1, c_2, b_1, b_3\}$ induces a diamond. So $b_1c_2, b_2c_2 \notin E$. Since $\{c_1, b_1, b_2, b_3\}$ cannot induce a diamond, b_3 is nonadjacent to at least one of b_1 and b_2. If $b_1b_3 \notin E$, then $b_3 \in B_1$, or else $\{b_1, c_1, c_2, b_3, a_i, a_j\}$ induces a P_6 for some i, j. Similarly, if $b_2b_3 \notin E$, then $b_3 \in B_2$. So by symmetry we may assume that $b_3 \in B_1$, $b_1b_3 \notin E$ and $b_2b_3 \in E$.

By Claim 24 we may assume that $b_4 \in B_3$ is nonadjacent to c_1. If $c_2b_4 \in E$, then $b_3b_4 \in E$. Then either $\{c_2, b_2, b_3, b_4\}$ induces a diamond, or $\{b_4, c_2, c_1, b_2, a_2, a_1\}$ induces a P_6, depending on whether b_2 and b_4 are adjacent. So $c_2b_4 \notin E$. Then $\{c_2, c_1, b_2, b_4, a_3, a_1\}$ or $\{c_2, c_1, b_2, a_2, a_3, b_4\}$ induces a P_6, depending on whether b_2 and b_4 are adjacent. □

Now suppose that $c_1 \in C$ has a neighbour in $b_1 \in B_1$, then c_1 has no neighbours in $B \setminus B_1$ by Claim 25. If c_1 has no neighbours in C, then $N_G(c_1) \subseteq N_G(a_1)$. So c_1 has a neighbour $c_2 \in C$. This implies that C has no trivial components. By Claim 25 we may assume that $b_2 \in B_2$ is nonadjacent to c_2. If $c_2b_1 \notin E$, then $\{c_2, c_1, b_1, b_2, a_2, a_3\}$ or $\{c_2, c_1, b_1, a_1, a_2, b_2\}$ induces a P_6, depending on whether b_1 and b_2 are adjacent. So $N_B(c_2) \subseteq N_B(c_1)$. Moreover by symmetry and transitivity, every vertex in a component K of C has the same neighbourhood in B. Then either $N_B(K)$ is a clique cutset, or $K \cup N_B(K)$ contains a diamond. So C is empty. □

5.2 Proof of Theorem 10

Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 4. Let $A = \{a_1, a_2, a_3, a_4\}$ be a K_4 in G, $B_i = \{v \in V \setminus A : va_i \in E\}$ for $i = 1, 2, 3, 4$, $B = \bigcup_{i=1}^{4} B_i$. By Lemma 6, G is $(K_4 + K_1)$-free. So $V = A \cup B$ and we may use the properties of B mentioned in Lemma 6.

Claim 26. *$B \setminus B_i$ is not 2-colourable for $i = 1, 2, 3, 4$.*

Proof. By symmetry suppose that $B_2 \cup B_3 \cup B_4$ is 2-colourable. Then G has a 5-colouring: colour a_1 with colour 1, $B_1 \cup \{a_2, a_3, a_4\}$ with colours in $\{2, 3, 4\}$, and $B_2 \cup B_3 \cup B_4$ with colours in $\{1, 5\}$. □

Claim 27. *If B_i contains a K_3, then B_j contains no K_3 for $j \neq i$.*

Proof. Let $K = \{b_1, b_2, b_3\}$ be a K_3 in B_1, and $L = \{b_4, b_5, b_6\}$ be a K_3 in B_2. Let $b_7 \in B_3$. Since G is (diamond,K_4+K_1)-free, b_7 has exactly one neighbour in K and exactly one neighbour in L, assume that $b_1b_7, b_4b_7 \in E$. At least one of b_5 and b_6 is nonadjacent to b_1, assume that $b_5b_1 \notin E$. One of b_2 and b_3 is nonadjacent to b_5, assume that $b_2b_5 \notin E$. Then $\{b_5, a_2, a_3, b_7, b_1, b_2\}$ induces a P_6. □

Claim 28. *B contains no C_5.*

Proof. We use some properties from Section 3 to prove this claim. Let $Q = \{v_1, v_2, v_3, v_4, v_5\}$ induce a C_5 in B with edges v_iv_{i+1} for $i = 1, \ldots, 5$, all indices modulo 5. By (P2) and (P21), $Q \subseteq B_i \cup B_j$. Assume that $v_1, v_3 \in B_1$, $v_2, v_4, v_5 \in B_2$.

Let u be an arbitrary vertex in $B_3 \cup B_4$. By symmetry let $u \in B_3$. If u has no neighbours in Q, then $\{u, a_3, a_1, v_1, v_5, v_4\}$ induces a P_6. By (P21), u is adjacent to two or three nonsequential vertices in Q. If u has exactly two neighbours v_i and v_{i+2} in Q, then $\{a_4, a_3, u, v_i, v_{i+4}, v_{i+3}\}$ induces a P_6. Since v_4 and v_5 are in the same component of B_2, $N_Q(u) \neq \{v_2, v_4, v_5\}$. So $N_Q(u) = \{v_1, v_2, v_4\}$ or $\{v_1, v_3, v_4\}$ or $\{v_1, v_3, v_5\}$ or $\{v_2, v_3, v_5\}$. Note that each B_i contains at least two vertices. Then by (P13), $B_3 \cup B_4$ is stable, and each of B_3 and B_4 contains exactly two vertices.

By Claim 26, $B_1 \cup B_3 \cup B_4$ contains a K_3 or a C_5. If $B_1 \cup B_3 \cup B_4$ contains a C_5, then B_2 contains exactly two vertices, a contradiction. So $B_1 \cup B_3 \cup B_4$ contains a K_3. Since there are no edges between B_3 and B_4, B_1 contains a K_3. Then $B_2 \cup B_3 \cup B_4$ contains no K_3 by Claim 27 and so contains a C_5. Then B_1 contains exactly two vertices, a contradiction. □

Claim 29. *For every component K of B_i of size 3, there is a K_4 in B containing one vertex in each of $B_1, \ldots, B_4$, and containing one vertex in K.*

Proof. Assume that $K = \{b_1, b_2, b_3\}$ is a K_3 in B_1. By Claim 26, Claim 27 and Claim 28, there are $b_4 \in B_2, b_5 \in B_3, b_6 \in B_4$ such that $L = \{b_4, b_5, b_6\}$ is a K_3. Since G is diamond-free, every vertex in K cannot have exactly two neighbours in L. If some vertex in K is complete to L, then we are done. So suppose that every vertex in K has at most one neighbour in L. Suppose that $b_1b_4 \in E$. By symmetry we may assume that $b_2b_5 \notin E$. Then $\{b_2, b_1, b_4, b_5, a_3, a_4\}$ induces a P_6. So K and L are anti-complete, and then $K \cup \{a_1, b_4\}$ induces a $K_4 + K_1$. $\square$

Claim 30. *If no B_j contains a K_3 for $j = 1, 2, 3, 4$, then for every component K of B_i of size 2, there is a K_4 in B containing one vertex in each of $B_1, \ldots, B_4$, and containing one vertex in K.*

Proof. Assume that no B_j contains a K_3 for $j = 1, 2, 3, 4$, and $K = \{b_1, b_2\}$ is a K_2 in B_1. By Claim 26 and Claim 28, there are $b_3 \in B_2, b_4 \in B_3, b_5 \in B_4$ such that $L = \{b_3, b_4, b_5\}$ is a K_3. Since G is diamond-free, every vertex in K cannot have exactly two neighbours in L. If some vertex in K is complete to L, then we are done. So suppose that every vertex in K has at most one neighbour in L. By a similar proof as for the proof of Claim 29 we have that K and L are anti-complete.

Since A is not a clique cutset, B is connected. Assume that $b_1, u_1, \ldots, u_n, b_3$ is a shortest path between K and L in B. Since K is a component of B_1, $u_1 \notin B_1$, and so $v_2u_1 \notin E$. Note that u_n is either complete to L, or nonadjacent to b_4, b_5 since G is diamond-free. If u_n is complete to L, then $L \cup \{u_n, b_2\}$ induces a $K_4 + K_1$. So $u_nb_4, u_nb_5 \notin E$. If $n \geq 2$, then $\{b_2, b_1, u_1, \ldots, u_n, b_3, b_4\}$ contains a P_6. So $n = 1$. If $u_1 \in B_2$, then $\{b_2, b_1, u_1, b_3, b_4, a_3\}$ induces a P_6. If $u_1 \notin B_2$, by symmetry assume that $u_1 \notin B_3$, then $\{b_2, b_1, u_1, b_3, a_2, a_3\}$ induces a P_6. $\square$

If every B_i is stable for $i = 1, 2, 3, 4$, then G is 4-colourable. So one of $B_1, \ldots, B_4$ contains a component of size 3 or 2. By Claim 29 and Claim 30, we may assume that $A' = \{b_1, b_2, b_3, b_4\}$ is a K_4 in B such that $b_i \in B_i$ for $i = 1, 2, 3, 4$, and there is a vertex v_1 adjacent to a_1 and b_1.

Since G is ($K_4 + K_1$,diamond)-free, every vertex in $V \setminus (A \cup A')$ is adjacent to exactly one vertex in A and exactly one vertex in A'. Let v_2 be a neighbour of v_1 in $V \setminus (A \cup A')$. Because G is diamond-free, v_2 is adjacent to a_1 if and only if it is adjacent to b_1. By $\omega(G) = 4$, v_1 has at most one neighbour in $V \setminus (A \cup A')$ which is adjacent to a_1 and b_1. Now assume that v_2 is nonadjacent to a_1. There are two cases up to symmetry: $v_2a_2, v_2b_2 \in E$ or $v_2a_2, v_2b_3 \in E$. In either case, $\{b_4, b_1, v_1, v_2, a_2, a_3\}$ induces a P_6. So v_1 has no neighbours in $V \setminus (A \cup A')$ which are nonadjacent to a_1. Then $d(v_1) \leq 3$, a contradiction. So there are no 6-vertex-critical (P_6,diamond)-free graphs with clique number 4.

5.3 Proof of Theorem 11

Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 5. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a K_5 in G, $B_i = \{v \in V \setminus A : va_i \in E\}$ for $i = 1, 2, 3, 4, 5$, $B = \bigcup_{i=1}^{5} B_i$. By Lemma 6, G is $(K_5 + K_1)$-free. So $V = A \cup B$ and we may use the properties of B mentioned in Lemma 6.

Claim 31. *If B_i contains a K_3, then B_j contains no K_3 for $j \neq i$.*

Proof. Let $K = \{b_1, b_2, b_3\}$ be a K_3 in B_1, and $L = \{b_4, b_5, b_6\}$ be a K_3 in B_2. Now we prove that $K \cup L$ and $B \setminus (B_1 \cup B_2)$ are anti-complete. Suppose that $b_7 \in B_3$ is adjacent to b_1, then $b_7b_2, b_7b_3 \notin E$. Since each of b_1 and b_7 has at most one neighbour in L, assume that $b_4b_1, b_4b_7 \notin E$. We may assume that $b_2b_4 \notin E$ by symmetry. Then $\{b_4, a_2, a_3, b_7, b_1, b_2\}$ induces a P_6. So $K \cup L$ and $B \setminus (B_1 \cup B_2)$ are anti-complete.

Since A is not a clique cutset, B is connected. Assume that $b_1, u_1, \ldots, u_n, b_8$ is a shortest path between $K \cup L$ and $B \setminus (B_1 \cup B_2)$ in B. By symmetry assume that $b_8 \in B_3$. Then $u_1, \ldots, u_n \in B_1 \cup B_2$. If $b_2u_1 \notin E$, then $\{b_2, b_1, u_1, \ldots, u_n, b_8, a_3, a_4\}$ contains a P_6. So $b_2u_1 \in E$, and then $K \cup \{u_1\}$ is a K_4 in B_1. Then $n = 1$, or else $K \cup \{u_1, a_1, b_8\}$ induces a $K_5 + K_1$. Since each of b_1 and u_1 has at most one neighbour in L, assume that $b_4b_1, b_4u_1 \notin E$. Then $\{b_1, u_1, b_8, a_3, a_2, b_4\}$ induces a P_6. □

Claim 32. *Every B_i is K_4-free.*

Proof. Suppose that $K = \{b_1, b_2, b_3, b_4\}$ is a K_4 in B_1. If $B \setminus B_1$ is stable, then G is 5-colourable. So let $L = \{b_5, b_6\}$ be a K_2 in $B \setminus B_1$, then each of b_5 and b_6 has exactly one neighbour K.

First we assume that b_5 and b_6 are in the same B_i, say B_2. Then their neighbours in K are different. Figure 9 is a diagram of the following proof. Assume that $b_1b_5, b_2b_6 \in E$. Suppose that b_5 has a neighbour $b_7 \in B_3$. Then $b_6b_7 \notin E$. At least one of b_3 and b_4 is nonadjacent to b_7, assume that $b_3b_7 \notin E$. Then $\{b_6, b_5, b_7, a_3, a_1, b_3\}$ induces a P_6. So b_5 has no neighbours in $B_3 \cup B_4 \cup B_5$. By Claim 31, since B_1 contains a K_4, b_5 has only one neighbour b_6 in B_2. By $d(b_5) \geq 5$, b_5 has a neighbour $b_8 \in B_1 \setminus K$. Then $b_8b_6 \notin E$. By $d(b_3) \geq 5$, b_3 has a neighbour $b_9 \in B \setminus B_1$. Then $b_5b_9, b_6b_9 \notin E$. If $b_9b_8 \notin E$, then $\{b_8, b_5, b_6, b_2, b_3, b_9\}$ induces a P_6. If $b_9b_8 \in E$, then $\{b_8, b_9, a_2, b_6, b_2, b_4\}$ or $\{b_8, b_9, b_3, b_2, b_6, a_2\}$ induces a P_6, depending on whether $b_9 \in B_2$. So b_5 and b_6 cannot be in the same B_i.

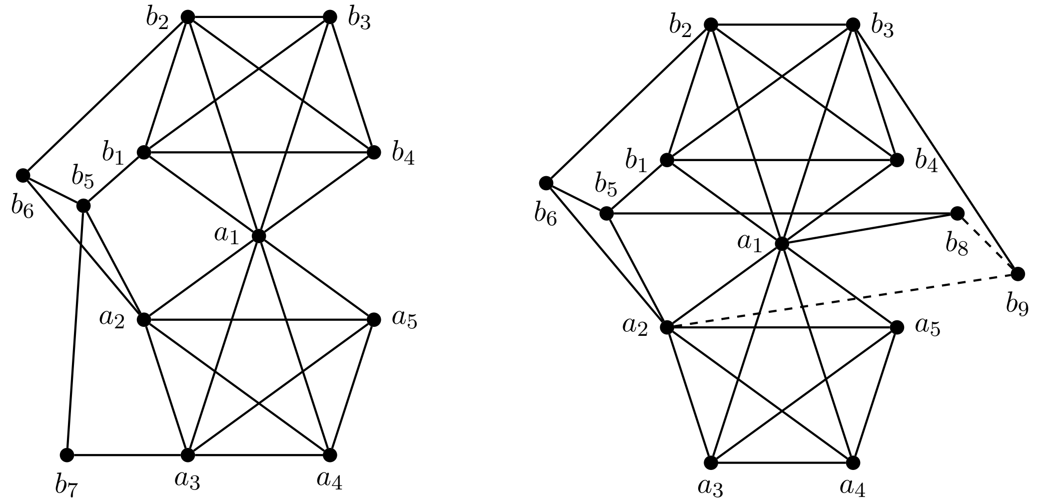

Figure 9: Diagram for Claim 32. Some unimportant uncertain edges are not drawn.

Now we assume that $b_5 \in B_2$, $b_6 \in B_3$ by symmetry. Let $b_1b_5 \in E$ and $b_2b_6 \notin E$. If $b_1b_6 \notin E$, then $\{b_2, b_1, b_5, b_6, a_3, a_4\}$ induces a P_6. So $b_1b_6 \in E$. Note that $K \cup \{a_1\}$ is a K_5, while $A \setminus \{a_1\}$ is a K_4 in the neighbourhood of a_1 and L is a K_2 in the neighbourhood of b_1. Then we can get a contradiction by symmetry. □

Claim 33. *B contains no C_5.*

Proof. Let Q=$\{v_1, v_2, v_3, v_4, v_5\}$ induce a C_5 in B with edges v_iv_{i+1} for $i = 1, \ldots, 5$, all indices modulo 5. By the properties from Section 3, we may assume that $v_1, v_3 \in B_1$, $v_2, v_4, v_5 \in B_2$. Then for any $u \in B_3 \cup B_4 \cup B_5$, $N_Q(u) = \{v_1, v_2, v_4\}$ or $\{v_1, v_3, v_4\}$ or $\{v_1, v_3, v_5\}$ or $\{v_2, v_3, v_5\}$. And then $B_3 \cup B_4 \cup B_5$ is stable and contains at most 4 vertices whose neighbourhoods in Q are pairwise different. Let $b_1 \in B_3$, $b_2 \in B_4$. Note that there exist $v_i, v_j \in Q$ such that $\{b_1, v_i, v_j, b_2\}$ induces a P_4. Then $\{a_5, a_3, b_1, v_i, v_j, b_2\}$ induces a P_6. □

Claim 34. *$B \setminus (B_i \cup B_j)$ is not 2-colourable for $i \neq j$.*

Proof. Suppose that $B_3 \cup B_4 \cup B_5$ is 2-colourable. Then $B_1 \cup B_2$ is not 3-colourable, or else G is 5-colourable. Since $B_1 \cup B_2$ contains no K_4 by Claim 32, $B_1 \cup B_2$ is imperfect. Then $B_1 \cup B_2$ contains a C_5, contradicting with Claim 33. $\square$

Since G is not 5-colourable, at least one of $B_1, \ldots, B_5$ is nonstable. By Claim 32, the largest component among $B_1, \ldots, B_5$ is of size 2 or 3.

Claim 35. *Let K in B_i be a largest component among $B_1, \ldots, B_5$. Then there is a K_4 containing one vertex in each of $K, B_{j_1}, B_{j_2}, B_{j_3}$ for every combination of three sets $B_{j_1}, B_{j_2}, B_{j_3}$ in $\{B_1, \ldots, B_5\} \setminus \{B_i\}$.*

Proof. Let $K = \{b_1, b_2\}$ or $\{b_1, b_2, b_3\}$ in B_1. By Claim 31, Claim 33 and Claim 34, there are $b_4 \in B_2, b_5 \in B_3, b_6 \in B_4$ such that $L = \{b_4, b_5, b_6\}$ is a K_3. If some vertex in K is complete to L, then we are done. So suppose that every vertex in K has at most one neighbour in L. Suppose that $b_1b_4 \in E$. By symmetry we may assume that $b_2b_5 \notin E$. Then $\{b_2, b_1, b_4, b_5, a_3, a_4\}$ induces a P_6. So K and L are anti-complete.

Since B is connected, assume that $\{b_1, u_1, \ldots, u_n, b_4\}$ is a shortest path between K and L in B. Note that $u_1 \notin B_1$, so $b_2u_1 \notin E$. If $n \geq 3$, then $\{b_2, b_1, u_1, \ldots, u_n, b_4\}$ contains a P_6. If $n = 2$, then u_2 is complete to L, for otherwise $\{b_2, b_1, u_1, u_2, b_4, b_5\}$ induces a P_6. By symmetry we may assume that $u_1, u_2 \notin B_2$. Then $\{b_2, b_1, u_1, u_2, b_4, a_2\}$ induces a P_6. So $n = 1$. If $u_1 \in B_2$, then $u_1b_5 \notin E$, or else $\{u_1, b_4, b_5, a_2\}$ induces a diamond. Then $\{b_2, b_1, u_1, b_4, b_5, a_3\}$ induces a P_6. If $u_1 \notin B_2$, by symmetry assume that $u_1 \notin B_3$, then $\{b_2, b_1, u_1, b_4, a_2, a_3\}$ induces a P_6. $\square$

Claim 36. *Let K be a largest component among $B_1, \ldots, B_5$, then there is a K_5 in B containing one vertex in each of $B_1, \ldots, B_5$, and containing one vertex in K.*

Proof. Let $K = \{b_1, b_2\}$ *or* $\{b_1, b_2, b_3\}$ in B_1. By Claim 35, there are $b_4 \in B_2$, $b_5 \in B_3$, $b_6 \in B_4$ such that $\{b_1, b_4, b_5, b_6\}$ is a K_4, and there is a vertex $b_7 \in B_5$ which has a neighbour in K. If b_7 is complete to $\{b_4, b_5, b_6\}$, then $b_7b_1 \in E$ and so we are done. Suppose that b_7 has at most one neighbour in $\{b_4, b_5, b_6\}$. By symmetry assume that $b_7b_4, b_7b_5 \notin E$.

If b_7 is adjacent to b_2 or b_3, by symmetry assume that $b_7b_2 \in E$, then $\{b_7, b_2, b_1, b_4, a_2, a_3\}$ induces a P_6. So b_7 is adjacent to b_1. Then $b_7b_6 \notin E$, or else $\{b_1, b_7, b_6, b_5\}$ induces a diamond. Then there are $b_8 \in B_2$ and $b_9 \in B_3$ such that $\{b_1, b_7, b_8, b_9\}$ is a K_4. Since $\{b_6, b_7, b_8, b_9\}$ cannot induce a diamond, at least one of b_8 and b_9 is nonadjacent to b_6, by symmetry assume that $b_8b_6 \notin E$. Since $\{b_8, b_4, b_5, b_6\}$ cannot induce a diamond, at least one of b_4 and b_5 is nonadjacent to b_8. If $b_8b_4 \notin E$, then $\{b_8, b_7, a_5, a_4, b_6, b_4\}$ induces a P_6. If $b_8b_5 \notin E$, then $\{b_8, b_7, a_5, a_4, b_6, b_5\}$ induces a P_6. $\square$

Now we may assume that $A' = \{b_1, b_2, b_3, b_4, b_5\}$ is a K_5 in B such that $b_i \in B_i$ for $i = 1, 2, 3, 4, 5$, and there is a vertex v_1 adjacent to a_1 and b_1. Similarly as in the proof of Theorem 10 we consider the neighbourhood of v_1. Then $d(v_1) \leq 4$, a contradiction. So there are no 6-vertex-critical (P_6,diamond)-free graphs with clique number 5.

6 Conclusion

In this paper, we improved the χ-bound of (P_6, diamond)-free graphs from $\chi(G) \leq \omega(G) + 3$ [7] to $\chi(G) \leq \max\{6, \omega(G)\}$. Moreover, we proved that the chromatic number of graphs in the class of (P_6, diamond)-free graphs can be calculated in polynomial time. We suspect that similar results can be obtained for other hereditary graph families: if a hereditary graph family has a χ-bound in the form of $\chi(G) \le \omega(G) + C$ where C is a constant, then it may be possible to improve the bound to $\max\{C', \omega(G)\}$ where C' is a constant. If that is the case, it may also be possible to compute the chromatic number in polynomial time for this family of graphs. However, this is not always possible since determining the chromatic number is NP-hard for the hereditary family of line graphs, which has a χ-bound in the form of $\chi(G) \le \omega(G) + C$. So it would be interesting to find other hereditary graph families for which the chromatic number can be determined in polynomial time in this way.

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Appendix

A Computer-Free Proof of a Weaker Version of Theorem 9

The proof of Theorem 9 uses computational methods. In this section we give a computer-free proof of a weaker version of Theorem 9. This weaker theorem still suffices to give a complete computer-free proof of Theorem 7 and Theorem 8.

Theorem 12. *There are finitely many 6-vertex-critical (P_6,diamond)-free graphs with clique number 3.*

Proof of Theorem 12. Let $G = (V, E)$ be a 6-vertex-critical (P_6,diamond)-free graph with clique number 3. Let $A = \{a_1, a_2, a_3\}$ be a K_3 in G, $B_i = \{v \in V \setminus A : va_i \in E\}$ for $i = 1, 2, 3$, $B = \bigcup_{i=1}^{3} B_i$. By Lemma 6, G is $(K_3 + K_1)$-free. So $V = A \cup B$ and we may use the properties of B mentioned in Lemma 6.

Claim 37. *$B \setminus B_i$ is K_3-free and not 3-colourable for $i = 1, 2, 3$.*

Proof. By symmetry assume that $i = 3$. Since G is K_4-free, each B_j cannot contain a K_3. If $\{b_1, b_2, b_3\}$ is a K_3 in $B_1 \cup B_2$, by symmetry assume that $b_1, b_2 \in B_1$, $b_3 \in B_2$, then $\{b_1, b_2, b_3, a_1\}$ induces a diamond. So $B_1 \cup B_2$ is K_3-free. If $B_1 \cup B_2$ is 3-colourable, then we have a 5-colouring for G: colour a_1, a_2, a_3 with colours $1, 2, 3$ in order, colour B_3 with colours in $\{1, 2\}$ and $B_1 \cup B_2$ with colours in $\{3, 4, 5\}$. So $B_1 \cup B_2$ is not 3-colourable. □

Claim 38. *$B \setminus B_i$ is an induced subgraph of the Clebsch graph for $i = 1, 2, 3$.*

Proof. By symmetry let $i = 3$. By Theorem 4 and Claim 37, every $B_1 \cup B_2$ contains the Grötzsch graph. Let K be a component of $B_1 \cup B_2$ such that K contains the Grötzsch graph. Let L be a graph obtained from K by successively deleting the vertex with the smaller neighbourhood of every pair of comparable vertices (if the neighbourhoods are equal, pick any of the two vertices). Let u_L be the last deleted vertex, if it exists. Then L contains the Grötzsch graph, and is an induced subgraph of the Clebsch graph by Theorem 4. Let $L = \{v_0, \ldots, v_{10}, \ldots\}$, whose indices correspond to Figure 1. Note that each of $v_{11}, \ldots, v_{15}$ may either exist or not exist.

Now we prove that every vertex in $(B_1 \cup B_2) \setminus L$ has a neighbour in L. Suppose that $u \in (B_1 \cup B_2) \setminus L$ has no neighbours in L. Since B_1 and B_2 are P_3-free, every induced C_5 in $B_1 \cup B_2$ has the property that two nonadjacent vertices of the C_5 belong to one of B_1 and B_2, and the other three vertices belong to the other of B_1 and B_2. Now we consider $\{v_6, v_7, v_8, v_9, v_{10}\}$, by symmetry we may assume that $v_6, v_8 \in B_1$, $v_7, v_9, v_{10} \in B_2$. Then $v_5 \in B_1$. Then $\{u, v_9, v_{10}, a_2\}$ or $\{u, v_5, v_6, a_1\}$ induces a $K_3 + K_1$, depending on whether u belongs to B_1. So every vertex in $(B_1 \cup B_2) \setminus L$ has a neighbour in L. This implies that $B_1 \cup B_2$ is connected.

Now assume that u_L exists, and by symmetry assume that $u_L \in B_1$. Since u_L is the last deleted vertex when we obtain L, there is a vertex $u_1 \in L$ such that $N_L(u_L) \subseteq N_L(u_1)$. There is a vertex $u_2 \in N_L(u_1)$ such that $u_L u_2 \in E$. Note that every K_2 in L is in a C_5 in L. Let $\{u_1, u_2, u_3, u_4, u_5\}$ be an arbitrary C_5 in L containing u_1 and u_2, such that $v_i v_{i+1} \in E$ for $i = 1, \ldots, 5$, all indices modulo 5. Now we prove that $u_L u_5 \notin E$. Suppose that $u_L u_5 \in E$. Now we discuss two cases, depending on u_1.

Case 1. $u_1 \in B_2$.

Since B_1 and B_2 are P_3-free, u_2 and u_5 belong to B_1 and B_2, respectively, by symmetry assume that $u_2 \in B_1$, $u_5 \in B_2$. Note that for every P_3 in L, there is a vertex in L which is anti-complete to that P_3. Let $v_i \in L$ be a vertex which is anti-complete to $\{u_2, u_1, u_5\}$, then $v_i u_L \notin E$ since $N_L(u_L) \subseteq N_L(u_1)$. Then $\{a_2, u_1, u_5, v_i\}$ or $\{a_1, u_L, u_2, v_i\}$ induces a $K_3 + K_1$.

Case 2. $u_1 \in B_1$.

Since u_L and u_1 are not comparable in G, $N_{A \cup B_3}(u_L) \nsubseteq N_{A \cup B_3}(u_1)$. Let $b \in B_3$ such that $bu_L \in E$, $bu_1 \notin E$. Since B_1 and B_2 are P_3-free, $u_2 u_5 \in B_2$, and at least one of u_3 and u_4 is in B_1. If $u_3, u_4 \in B_1$, then one of u_3 and u_4 is adjacent to b, or else $\{a_1, u_3, u_4, b\}$ induces a $K_3 + K_1$. If one of u_3 and u_4 is in B_2, say u_3, then one of u_2 and u_3 is adjacent to b, or else $\{b, a_3, a_1, u_1, u_2, u_3\}$ induces a P_6. So b is adjacent to at least one of u_2, u_3, u_4, u_5. Suppose that

$bu_2 \in E$. Then each vertex in $V \setminus \{b, u_L, u_2\}$ is adjacent to exactly one vertex in $\{b, u_L, u_2\}$. So $bu_4 \in E$, $bu_3, bu_5 \notin E$. Suppose that $bu_3 \in E$, then we have $bu_2 \notin E$. Then $bu_4 \notin E$, or else $\{b, u_3, u_4, u_1\}$ induces a $K_3 + K_1$. Then $bu_5 \in E$, or else $\{b, a_3, a_1, u_1, u_5, u_4\}$ induces a P_6. So by symmetry we may assume that $bu_2, bu_4 \in E$, $bu_3, bu_5 \notin E$. Note that for every K_2 in L, there is another K_2 which is anti-complete to the former K_2. Let $\{v_i, v_j\}$ be a K_2 in L which is anti-complete to $\{u_1, u_2\}$. Since every vertex in $V \setminus \{u_2, u_L, b\}$ has exactly one neighbour in $\{u_2, u_L, b\}$, we have $v_ib, v_jb \in E$. Then $\{b, v_i, v_j, u_1\}$ induces a $K_3 + K_1$.

So $u_Lu_5 \notin E$. Note that every P_3 in L is in a C_5 in L. Since $\{u_1, u_2, u_3, u_4, u_5\}$ is an arbitrary C_5 in L containing u_1 and u_2, u_L has only one neighbour u_2 in L. Note that for every vertex v in L, there is a P_5 in L such that v is an end of the P_5. Then $L \cup \{u_L\}$ contains a P_6. So u_L does not exist. This implies that $B_1 \cup B_2$ contains no comparable vertices. Then $B_1 \cup B_2 = L$ is an induced subgraph of the Clebsch graph. □

Since every $B \setminus B_i$ contains at most 16 vertices, G contains at most 27 vertices. So there are only finitely many 6-vertex-critical (P_6,diamond)-free graphs with clique number 3. □