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**Direct and inverse problems for singular partial  
differential equations with fractional order  
integral-differential operators**

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## Summary

The physical problems involving several variables are mathematically expressed by partial differential equations with specific conditions known as initial or boundary conditions. In this case, for mathematicians, it is interesting to study the existence and uniqueness of a solution, stability and so on.

In this thesis, we are concerned with studying the boundary value problems for fractional partial differential equations (PDEs), particularly, mixed equations involving the subdiffusion involving hyper-Bessel fractional differential operator in Caputo sense and classical wave equation or sometimes wave with fractional order derivative. The application of these types of equations might appear in the problems of aerodynamics and hydrodynamics in terms of presenting transonic flow and also, these equations are used to study gas flow with nearly sonic speeds. We investigate the existence and uniqueness of a solution in every case. Our main technique is to find the main functional relations from both domains and come to the ordinary differential equations and/or Fredholm integral equations. Moreover, the method of separation of variables is mostly used and using the completeness property of the system of eigenfunctions is applied in many cases. The problems we have solved have differences from each other in the type of fractional differential operator, boundary conditions, domains on which the problems are considered, and the methods for proving the uniqueness of a solution.

This dissertation consists of 4 chapters. In the first chapter, the auxiliary results and the mathematical background are presented. Subsequent chapters are based on 6 papers published in well-respected journals.

In the second chapter, we considered two problems, particularly, the boundary value problem and the nonlocal problem for the mixed equation involving subdiffusion and classical wave equation. In both problems, the inte-

gral energy method is used for proving the uniqueness of a solution. In order to show the existence, we derived the Fredholm integral equation which comes from main functional relations. The entire chapter is based on these two papers [119], [120] published in journals “Bulletin of the Institute of Mathematics” and “Montes Taurus Journal of Pure and Applied Mathematics”.

Chapter 3 is devoted to the study of 3 nonlocal problems for mixed equations involving with one-dimensional subdiffusion and fractional wave equations. The equations differ from each other with the fractional differential operators, the nonlocal conditions and domains are also taken differently. The main approach is the method of separating variables and the uniqueness of a solution is based on the completeness of the system of eigenfunctions. All these three problems are published with full details in the journals “Mathematical Methods in the Applied Sciences” [122] (joint work with M. Ruzhansky and E. Karimov), “Fractional Differential Calculus” [123], “Uzbek Mathematical Journal” [121] (joint work with E. Karimov).

The last chapter deals with the two direct and inverse problems for fractional pseudo-parabolic equations. In section 4.1, the nonlocal problem for the Langevin-type equation is discussed. The solution is found in form of Fourier-Legendre series. With the help of the properties of Legendre polynomials, we showed the existence of the solution. The result is published in the journal “International Journal of Applied Mathematics” [124] (joint work with E. Karimov). In addition, the inverse problem determining the time-dependent source term by means of an additional energy measurement is also considered for the same Langevin-type equation. The next section in this chapter discusses the solvability (uniqueness and existence) of direct and inverse problems for the pseudo-parabolic equation for 2D Landau Hamiltonian defined on the plane. Using the global Fourier analysis the theorems of the uniqueness and

existence of generalized solutions to the direct and inverse problems are proved. For the inverse problem (determination of a single location-dependent source based on a measurement at the final time) the stability analysis of the solution is also considered. In all problems, we have used the completeness properties of the system of eigenfunctions in order to show the uniqueness of a solution. The paper containing these results has been submitted for publication to the “Georgian Mathematical Journal”.



## **Samenvatting**

Samenvatting Fysische problemen worden vaak gemodelleerd met behulp van partiële differentiaalvergelijkingen (PDVn) en aangevuld met bijkomende voorwaarden die bekend staan als begin- of randvoorwaarden. Voor wiskundigen is het dan bijvoorbeeld interessant om het bestaan, de uniciteit en stabiliteit van een oplossing te bestuderen.

De doelstelling van het proefschrift is om gemengde vraagstukken, die bestaan uit diffusie- en golfvergelijkingen met fractionele-orde afgeleide geformuleerd op aangrenzende (tijds)domeinen, te onderzoeken. Deze problemen hebben toepassingen in aerodynamica en hydrodynamica. Bovendien worden deze vergelijkingen ook bijvoorbeeld gebruikt om een gasstroom met (bij benadering) transsonische snelheid te modelleren. De gemengde vergelijkingen bestaan uit een subdiffusie vergelijking (een fractionele warmtevergelijking met hyper-Bessel fractionele operator in de zin van Caputo) en een klassieke golfvergelijking of golfvergelijking met fractionele orde. Voor elk vraagstuk onderzoeken we het bestaan en de uniciteit van een oplossing. De oplossingsmethode bestaat eruit om representaties van de oplossingen op beide domeinen te vinden, en via bijkomende voorwaarden het probleem te reduceren tot een gewone differentiaalvergelijkingen en/of Fredholm-integraalvergelijkingen. Hierbij maken we gebruik van de methode van scheiding van veranderlijken en de eigenschap i.v.m. volledigheid van de eigenfuncties. De problemen die we hebben opgelost in de proefschrift kunnen van elkaar onderscheiden worden door het type van de fractionele differentiaaloperator, de beschouwde randvoorwaarden, de domeinen waarop de problemen worden geformuleerd en de methoden om de uniciteitsresultaten te bewijzen.

Dit proefschrift bestaat uit vier hoofdstukken. In Hoofdstuk 1 worden de hulpresultaten en de wiskundige achtergrond van het onderzoek geschetst.

Hoofdstukken 2-4 bevatten de resultaten van zes publicaties in gerespecteerde tijdschriften. De dissertatie wordt afgesloten met een samenvatting van de belangrijkste resultaten en enkele perspectieven voor verder onderzoek. In Hoofdstuk 2 bestuderen we twee gemengde vraagstukken bestaande uit een eendimensionale subdiffusie- en (klassieke) golfvergelijking, met respectievelijk klassieke en niet-lokale randvoorwaarden. De gemaakte analyse is analoog aan de gevolgde aanpak in Hoofdstuk 1. Ten eerste is de energiemethode gebruikt om de uniciteit van een oplossing aan te tonen. Ten tweede wordt het bestaan van een oplossing bekomen gebruik makend van de theorie van de Fredholm integraalvergelijkingen. Hoofdstuk 2 is gebaseerd op artikels [119], [120] gepubliceerd in de tijdschriften “Bulletin of the Institute of Mathematics” en “Montes Taurus Journal of Pure and Applied Mathematics”.

Hoofdstuk 3 is gewijd aan de studie van drie niet-lokale gemengde vraagstukken met eendimensionale subdiffusie- en fractionele golfvergelijkingen. De problemen onderscheiden zich van elkaar door het beschouwen van verschillende fractionele differentiaaloperatoren, niet-lokale randvoorwaarden en domeinen. Opnieuw is de oplossingsmethode gebaseerd op de methode van scheiding van veranderlijken, terwijl de uniciteit van een oplossing nu bouwt op de volledigheid van het stelsel van eigenfuncties. De inhoud van dit hoofdstuk is gebaseerd op drie artikels gepubliceerd in de tijdschriften “Mathematical Methods in the Applied Sciences” [122], “Fractional Differential Calculus” [123], “Uzbek Mathematical Journal” [121].

Hoofdstuk 4 behandelt directe en inverse problemen voor fractionele pseudo-parabolische vergelijkingen. Eerst bestuderen we het goed gesteld zijn van een niet-lokale eendimensionale Langevin PDV. De oplossing wordt gerepresenteerd als een Fourier-Legendre reeks. Met behulp van de eigenschappen van Legendre polynomen tonen we het bestaan van een (unieke) oplossing aan. Het

resultaat van dit onderzoek is gepubliceerd in het tijdschrift “International Journal of Applied Mathematics” [124]. Daarna focussen we op de reconstructie van een louter tijdsafhankelijke bron in dezelfde Langevin PDV door middel van een bijkomende energiemeting. Dit is een voorbeeld van een invers bronprobleem. De volgende sectie in dit hoofdstuk onderzoekt de oplosbaarheid (existentie en uniciteit) van directe en inverse problemen voor de pseudo-parabolische Landau Hamiltoniaan vergelijking, gedefinieerd in het vlak. Voor het inverse vraagstuk (bepaling van een enkel plaatsafhankelijke bron op basis van een meting op het eindtijdstip) wordt ook een stabiliteitsanalyse op de oplossing uitgevoerd. In alle problemen gebruiken we de volledigheidseigenschap van het stelsel van eigenfuncties om de existentie van een unieke oplossing te bekomen. Het artikel met deze resultaten is ingediend voor publicatie bij het “Georgian Mathematical Journal”.

# Introduction

**Actuality and demand of the theme of the dissertation.** The theory of partial differential equations (PDEs) is one of the important branches of mathematics which can be used to mathematically formulate and solve the physical and other problems involving functions of several variables, such as the fluid flow, propagation of heat or sound, elasticity, electrostatics, electrodynamics, etc [12]. Boundary-value, initial-value and initial-boundary value problems are 3 major types of problems for PDEs. We also note that PDEs can be categorized according to the type as hyperbolic, parabolic and elliptic, and mixed type PDE such as parabolic-hyperbolic, hyperbolic-elliptic or parabolic-elliptic [14], [92].

The most fundamental results on mixed-type PDEs were presented by Chaplygin [20] and the following equation is also called by his name which is closely connected with the theory of gas flow:

$$K(y)u_{xx} + u_{yy} = 0.$$

In order to show the importance of the mixed-type equation in studying transonic flow let us consider a two-dimensional adiabatic potential flow of a perfect gas [93]. The stream function  $u = u(x, y)$  satisfies

$$(\rho^2 \alpha^2 - u_y^2) u_{xx} + 2u_x u_y u_{xy} + (\rho^2 \alpha^2 - u_x^2) u_{yy} = 0, \quad (*)$$

where  $\alpha$  is the local velocity of sound,  $\rho$  is the density of the gas.

We use the hodograph transformation  $\varphi = \rho^{-1}u_y$ ,  $\psi = -\rho^{-1}u_x$  and here  $\varphi, \psi$  are the rectangular velocity components as new independent variables. We consider the polar coordinates with respect to the corresponding components:

$$r = \sqrt{\varphi^2 + \psi^2}, \quad \theta = \tan^{-1} \left( \frac{\psi}{\varphi} \right).$$

Now we introduce new independent variable  $t = \left( \frac{r}{r_0} \right)$  in order to normalize  $r$ , where  $r_0$  is the speed corresponding to zero density, then the above equation (\*) becomes

$$\frac{\partial}{\partial t} \left\{ \frac{2t}{(1-t)^\beta} u_t \right\} + \frac{1 - (1+2\beta)t}{2(1-t)^{\beta+1}} u_{\theta\theta} = 0,$$

where  $\beta = c_\psi/(c_p - c_\psi)$ , such that  $c_p$  denotes the specific heat at a constant pressure,  $c_\psi$  is the specific heat at a constant volume and  $r_0 = k\gamma\rho_0^{-1}/(\gamma - 1)$ ,  $\gamma = c_p/c_\psi$ ,  $\rho_0$  is the density of gas at zero speed,  $k = \text{const}$ , such that  $p = k\rho^\gamma$ .

We introduce the following notations

$$\xi = \theta, \quad \eta = - \int_{\frac{1}{2\beta+1}}^t \frac{(1-s)^\beta}{2s} ds.$$

Then, by applying this transformation to the last equation, we obtain the following Chaplygin equation

$$K(\eta)u_{\xi\xi} + u_{\eta\eta} = 0, \tag{**}$$

where  $K(\eta) = \frac{1 - (1+2\beta)t}{(1-t)^{1+2\beta}}$ .

In addition,  $K(0) = 0$ , that is  $\eta = 0$  for  $t = \frac{1}{2\beta+1}$ , in this case the velocity is equal to the local velocity of sound, and therefore above equation type (\*\*) will be parabolic.

Moreover,  $K(\eta) > 0$ , for  $t < \frac{1}{2\beta+1}$ ,  $\eta > 0$ , corresponding to subsonic velocities, and equation (\*\*) will be elliptic.

Furthermore,  $K(\eta) < 0$ , because  $\eta < 0$  for  $t > \frac{1}{2\beta+1}$ , corresponding to supersonic velocities, and equation (\*\*) becomes hyperbolic. Therefore, the

PDE (\*\*) is called as mixed type. This is one of the examples which shows the importance of studying mixed type PDEs in applied sciences.

At the same time it is worth saying, many real-world problems can be better explained with fractional operators than using integer order calculus. The investigations of real-life problems based on solving fractional differential equations have been gaining the considerable popularity and importance because of their an adequate applications in many fields of science, engineering in the last few decades [113], [80], [77].

In terms of considering the boundary-value problems for PDEs, we can also divide them into two big classes, i.e. direct and inverse problems. The direct problems are usually specialized in finding solutions, however if it is required to determine the coefficients or right-hand side (the source term, in the case of the heat equation, for example), initial or boundary condition, order of the equation (in fractional order case) additionally to the sought solution based on some additional given data, then such problems are known as inverse problems. Inverse problems may appear in various areas of human activity, for instance, biology, medicine, mineral exploration, quality control of industrial goods and others [50]. The references [104], [8] devoted to studying inverse source problems for linear and nonlinear PDEs are worth noting, which presented some exciting results.

In this dissertation we mostly dealt with investigating direct and inverse problems for mixed equations with fractional order derivatives which can be more general than aforementioned problems. An interesting and distinctive feature of the fractional calculus is that it is possible to present different definitions of fractional order integrals and derivatives; furthermore, many instances of those definitions are being applied and discussed to analyze certain processes [109]. As an example, we can take the Riemann-Liouville and Caputo fractional

order integral-differential operators which have been used widely to describe mathematical models of the many natural phenomena (see the references [91], [4], [84], [32], [85], [9]). In our case, we have used more general fractional differential operators such as the bi-ordinal Hilfer fractional differential operator and regularized Caputo-like counterpart of hyper-Bessel differential operator which are main tools to investigate further problems. In general, the bi-ordinal Hilfer fractional differential operator is preserved in terms of its interpolation concept, more precisely, when  $\mu = 0$ , it describes the Riemann-Liouville fractional derivative of  $\beta$  order and for  $\mu = 1$ , the bi-ordinal Hilfer derivative expresses the Caputo fractional derivative of order  $\alpha$ . Similarly, the regularized Caputo-like counterpart of hyper-Bessel differential operator is also generalizes the Caputo fractional differential operator in terms of singular case. Using these types of differential operators gives us a chance to obtain more general results which can be applicable in a wide broad of applications and helps to improve the quality of research.

**The degree of scrutiny of the problem:** In recent years studying boundary value problems for fractional order mixed PDEs become of interest in connection with their applications in many branches of mathematical physics [55]. Using more general fractional differential operators gives us a chance to improve the quality of research in terms of modeling real-life phenomena. Investigating the dynamics and movements of gas in the channel surrounded by a porous medium with the help of the mixed-type partial differential equations was proposed by I.M. Gelfand. He also showed that studying the propagation of vibrations in complicated electrical networks can be explained by means of mixed-type PDEs.

The beginning of effective investigations on mixed PDEs dates back to years in the last century. The interesting results were taken by Chaplygin,

Tricomi, Gelfand and others. Later, the fundamental results for parabolic-hyperbolic equations were developed by Salahitdinov, Juraev, Nakhushev, Vragov, and Kapustin. Currently, a huge number of works have been carrying out for fractional and integer parabolic-hyperbolic PDEs. In particular, Sh. Alimov, R. Ashurov, J. Tokhirov, A. Berdyshev, A. Pskhu, K. Sabitov, M. Sadybekov, E. Karimov, A. Urinov, Y. Apakov, A. Nagornyy and others studied local and nonlocal boundary value problems for parabolic-hyperbolic equations with a characteristic line of type changing and investigated the spectral properties of various boundary value problems for such equations. In addition, proposing and studying various types of problems for PDEs involving the fractional order differential operators are being gained a great interests for mathematicians. In particular, M. Yamamoto, R. Ashurov, M. Slodicka, Y. Luchko, V. Kiryakova, B. Turmetov, B. Kadirkulov, N. Tokmagambetov, H.M. Srivastava, M. Kirane, E. Karimov, N. Salte, D. Durdiev, M. Ruzhansky and others are working on such equations.

Investigating inverse problems for PDEs involving fractional order differential operators is also important part of research carried out in mathematics because of its applications in science and engineering. While essential results have been obtained by R. Ashurov on determining the order of the PDEs in the inverse problems, the inverse source problem for finding time dependent source are investigated by M. Slodicka, S. Malik, A. Hazanee and others. Despite of those investigations, there are still some open problems which have interesting applications.

The obtained results came out from discussions between associative professor E.T. Karimov (Institute of Mathematics, Uzbekistan) and professor M. Ruzhansky (Ghent University, Belgium) during my visits for studying joint-PhD program in 2021-2022.



**The aim of research work:** To study the unique solvability of direct and inverse problems for singular PDEs with fractional order integral-differential operators.

**Research methods.** In the dissertation several methods have been applied. For example, in order to show the uniqueness of the result, we mainly address the completeness property of the system of eigenfunctions while the method of energy integrals is applicable. To show the existence of the result we mostly analyze the second kind of Fredholm integral equation. Furthermore, properties of Mittag-Leffler function and Fourier-Bessel, Fourier-Legendre series are applied, as well.

**Scientific novelty of the research work** is presented as follows: A new definition of a regularized Caputo-like counterpart of hyper-Bessel fractional differential operator with arbitrary starting point is introduced and the solution of Cauchy problem is presented;

A unique solvability of an analogy of the Tricomi problem for the mixed equation involving subdiffusion equation and wave equation is proved;

Frankl type problem for the mixed equation involving subdiffusion and wave equations is investigated for unique solvability;

An unique solution of the Cauchy-type problem for ordinary fractional differential equation with the right hand-sided bi-ordinal Hilfer fractional derivative is found explicitly;

The uniqueness and existence results are proved for a non-local boundary value problem which is formulated for mixed FPDEs;

Direct and inverse problems are investigated for the pseudo-parabolic equation involving with the bi-ordinal Hilfer fractional derivative and unique solvability of considered problems is proved.

**Scientific and practical significance of the research results.**

The results of this thesis are mainly of theoretical significance. The results have a scientific impact to improve the quality of research in studying the different types of problems for mixed PDEs. Moreover, we have used specific fractional differential operators which play an essential role to generalize these kinds of problems.

When it comes to the practical significance of the thesis, it can be a particular cases of several models of physical phenomena. For example, a gas movement in a channel surrounded by porous medium will be governed by the mixed parabolic-hyperbolic type equation, because inside of the channel movement will be described by the wave equation, in porous media by diffusion equation. Our results can be an example in terms of generalizing these types of problems related to these models. Moreover, the results can be used to develop the model which describes the behaviour of a quantum particle in two dimensions under the influence of a constant magnetic field.

**Approbation of the research results.** The main result of the dissertation were discussed at the following international and republic scientific conferences: “Modern problems of mathematics and informatics” (Fergana 2019), “Modern problems of differential equations and related branches of mathematics” (Fergana 2020), “Actual problems of Stochastic Analysis”, (Tashkent 2021), “Annual International April Mathematical Conference” (Almaty 2021), “Modern problems of applied mathematics and information technologies al-Khwarizmi 2021” (Fergana 2021), “Modern problems of applied mathematics and information technologies” (Bukhara 2022), “Annual International April Mathematical Conference-2022” (Almaty 2022)

This dissertation work was discussed at the republican seminars “Modern problem of mathematical physics” and the seminar of department of “Differential equations and their applications” of the Institute of Mathematics, Uzbek-

istan Academy of Sciences and at the seminar of “Namangan branch of Institute of Mathematics”.

**Publications of the research results.** On the topic of the dissertation, 13 scientific papers were published, 6 of which are included in the list of scientific publications proposed by the Higher Attestation Commission of the Republic of Uzbekistan for the defense of theses of the Doctor of Philosophy, including 4 of them published in foreign journals and 2 in national scientific journals and 7 abstracts.

**The structure and volume of the dissertation.** The dissertation consists of the introduction, 4 chapters, summary and bibliography. The total volume of the thesis is 156 pages.

# Chapter 1

## Preliminaries

In this chapter, we briefly give the necessary definitions, main properties of the fractional integral-differential operators which are used in further chapters. Also, we give main notations and a brief introduction to the fractional calculus, as well.

### 1.1 Fractional calculus - Short historical survey

Fractional calculus is as old as classical calculus with integer order of derivative. It dates back to 17 century when G. A. l'Hopital asked a question to G. W. Leibniz in order to know the meaning of the derivative  $\frac{d^n}{dx^n}$  when  $n = \frac{1}{2}$ . G. W. Leibniz carefully replied: "This is an apparent paradox from which, one day, useful consequences will be drawn...". This time is believed that an initial point of the fractional calculus.

Before presenting the definitions of fractional differential operators we use subsequent chapters, it is worth highlighting the important works (events) proposed by many mathematicians that have taken place so far. For example, G. W. Leibniz mentioned the term derivative of the general order in another letter to J. Bernoulli. In 1697, G. W. Leibniz wrote a letter to J. Wallis to discuss his infinite product for  $\pi$  and used a notation  $d^{\frac{1}{2}}y$  to signify the derivation of

order  $\frac{1}{2}$ . While P.S. Laplace wrote an expressions for derivative of non-integer order, S. F. Lacroix obtained formally the derivative of order  $\frac{1}{2}$ , arriving at the expression  $\frac{d^{1/2}}{dx^{1/2}}x = \frac{2\sqrt{x}}{\pi}$ . J. Liouville suggested the formula for differentiating of non-integer order in terms of infinite series in 1835. In 1855, he proposed a series of definitions for fractional derivative and then in 1873 he discussed the integration of differential equations with fractional order. B. Riemann also introduced an expression for the fractional integral with two changes comes to the expression we use today. For complete time line of events in this area during 1695-1970 see [75]

In short, during the last centuries, many mathematicians developed and generalized the idea of fractional integration and differentiation. For example, Caputo developed a definition, more appropriate than Riemann-Liouville derivative. Differentiating of non-integer order of the Caputo version is considered an important tool to discuss many problems involving fractional derivative.

In 1968, M. M. Dzhrbashian and A. B. Nersesyan published an important paper devoted to studying a general fractional differential operator similar to the Caputo operator. So far, a huge number of articles have been published devoted to studying Fractional Calculus which shows its importance in Mathematics. The all information of the development of the fractional calculus, from its origins until recent times, is taken from these references [75], [83], [78], [76], [19].

## 1.2 A brief introduction to functional spaces

Before moving on to the basic information about fractional integral and differential operators, let us consider some spaces in which above mentioned operators are defined. We refer [85], [66], [23] for further discussion in this direction.

Let  $\Omega = [a, b]$  be a finite or infinite interval on the real axis  $\mathbb{R}$ . The set

of Lebesgue complex-valued measurable functions on  $\Omega$  is denoted by  $L^p(a, b)$   $1 \leq p < \infty$  with the following norm

$$\|f\|_{L^p(a,b)} = \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

and

$$\|f\|_{L^\infty(a,b)} = \operatorname{ess\,sup}_{a \leq x \leq b} |f(x)|$$

where  $\operatorname{ess\,sup} |f(x)|$  is the essential maximum of the function  $|f(x)|$ ; for  $1 \leq p < \infty$  it forms a Banach space. When  $p = 2$ , the set of  $L^2(\Omega)$  is represented as a Hilbert space with the following scalar product

$$(f, g) = \int_a^b f(x)g(x)dx.$$

**Definition 1.2.1.** Let  $\Omega = [a, b]$  be a finite interval on  $\mathbb{R}$  and  $n \in \mathbb{N}$ . We denote by  $AC[a, b]$  the space of absolutely continuous functions on  $\Omega$ , i.e.,

$$f(x) \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt, \quad \varphi(t) \in L^1(a, b).$$

Also, we denote by  $AC^n[a, b]$  the space of complex-valued functions which have absolute continuous derivatives up to order  $n - 1$  on  $[a, b]$ , i.e.,

$$AC^n[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{C}, \quad f^{(n-1)}(x) \in AC[a, b] \right\}.$$

**Definition 1.2.2.** Let  $\Omega = [a, b]$  and  $m \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ . We introduce  $C^m(\Omega)$  the space of functions  $f$  which are  $m$  times continuously differentiable on  $\Omega$  defined by the following norm

$$\|f\|_{C^m(\Omega)} = \sum_{k=0}^m \left\| f^{(k)} \right\|_{C(\Omega)} = \sum_{k=0}^m \max_{x \in \Omega} \left| f^{(k)}(x) \right|.$$

We also use a weighted modification of the space  $C[a, b]$  ( $n \in \mathbb{N}_0$ ) which was introduced for the first time by Dimovski in his papers devoted to the operational calculus for the hyper-Bessel differential operator [23] for the first time in 1966:

$$C_\mu^{(n)} := \left\{ f(t) = t^p f_1(t), \quad f_1(t) \in C^{(n)}[0, \infty) \right\}, \quad C_\mu := C_\mu^{(0)} \text{ with } \mu \in \mathbb{R} \quad (1.2.1)$$

if there exists  $p > \mu$  for fixed  $\mu \geq -1$ . Clearly,  $C_\mu$  is a vector space and the set of spaces  $C_\mu$  is ordered by inclusion according to [49]

$$C_\mu \subset C_\delta \Leftrightarrow \mu \geq \delta.$$

In 2000, Kilbas et.al. presented [65] the weighted space  $C_\gamma[a, b]$  of functions  $g$  on  $[a, b]$  such that  $(t - a)^\gamma g(t) \in C[a, b]$  in the similar meaning:

$$C_\gamma[a, b] = \left\{ g(t) : \|g\|_{C_\gamma} = \|(t - a)^\gamma g(t)\|_C < \infty \right\}.$$

The weighted space  $C_\gamma^n[a, b]$  of functions  $g$  on  $[a, b]$  is defined by

$$C_\gamma^n[a, b] = \left\{ g : [0, T] \rightarrow \mathbb{R}, g^{(n-1)}(t) \in C[a, b]; g^{(n)}(t) \in C_\gamma[a, b] \right\}$$

with the norm

$$\|g\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \|g^{(k)}\|_{C[a, b]} + \|g^{(n)}\|_{C_\gamma[a, b]}.$$

The properties and the usage of these spaces are presented in the references devoted to studying the problems involving Hilfer fractional differential operator (see [66], [37]).

## 1.3 Fractional differential operators

### 1.3.1 The Riemann-Liouville and Caputo fractional differential operators

Now, we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval  $\Omega = [a, b]$  ( $-\infty < a < b < \infty$ ) of the

$\mathbb{R}$ . More detailed information might be found in the references [85], [66].

**Definition 1.3.1.** [66] *The left-hand sided Riemann-Liouville fractional integral  $I_{a+}^\alpha f(t)$  and the right hand-sided integral  $I_{b-}^\alpha f(t)$  of order  $\alpha > 0$  are defined by*

$$I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau) d\tau}{(t - \tau)^{1-\alpha}}, \quad t > a,$$

and

$$I_{b-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau) d\tau}{(\tau - t)^{1-\alpha}}, \quad t < b,$$

respectively, where  $\Gamma(\alpha)$  is Euler's gamma-function.

**Lemma 1.3.2.** [73], [74] *The Riemann-Liouville fractional integral  $I^\alpha$ ,  $\alpha \geq 0$ , is a linear map of the space  $C_\mu$ ,  $\mu \geq -1$ , into itself, that is*

$$I^\alpha : C_\mu \rightarrow C_{\alpha+\mu} \subset C_\mu.$$

**Definition 1.3.3.** [66] *The left-hand sided Riemann-Liouville fractional derivative  $D_{a+}^\alpha f(t)$  and the right hand-sided derivative  $D_{b-}^\alpha f(t)$  of order  $\alpha$  ( $n - 1 < \alpha \leq n$ ) defined as*

$$\begin{aligned} D_{a+}^\alpha f(t) &= \left( \frac{d}{dt} \right)^n I_{a+}^{n-\alpha} f(t) = \\ &= \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau \end{aligned}$$

and

$$\begin{aligned} D_{b-}^\alpha f(t) &= (-1)^n \left( \frac{d}{dt} \right)^n I_{b-}^{n-\alpha} f(t) = \\ &= \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau, \end{aligned}$$

These operators satisfy the following properties [66].



**Property 1.3.4.** If  $\alpha > 0$ ,  $\beta > 0$  and  $t \in [a, b]$ ,  $f(t) \in L^p(a, b)$ , ( $1 \leq p < \infty$ )

$$I_{a+}^\alpha I_{a+}^\beta f(t) = I_{a+}^{\alpha+\beta} f(t) \text{ and } I_{b-}^\alpha I_{b-}^\beta f(t) = I_{b-}^{\alpha+\beta} f(t).$$

**Property 1.3.5.** If  $f(t) \in L^1(a, b)$  and  $I_{a+}^{n-\alpha} f(t) \in AC^n[a, b]$ , then the inequality

$$I_{a+}^\alpha D_{a+}^\alpha f(t) = f(t) - \sum_{j=1}^n \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[ \lim_{t \rightarrow a+} \left( \frac{d}{dt} \right)^{n-j} I_{a+}^{n-\alpha} f(t) \right],$$

$$I_{b-}^\alpha D_{b-}^\alpha f(t) = f(t) - \sum_{j=1}^n \frac{(-1)^{n-j} (b-t)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[ \lim_{t \rightarrow b-} \left( \frac{d}{dt} \right)^{n-j} I_{b-}^{n-\alpha} f(t) \right]$$

hold almost everywhere on  $[a, b]$ .

**Property 1.3.6.** Let  $\alpha > 0$ ,  $p \geq 1$ ,  $q \geq 1$  and  $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$  ( $p \neq 1$ , and  $q \neq 1$  in the case when  $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$ ). If  $f(t) \in I_{b-}^\alpha(L^p)$  and  $g(t) \in I_{a+}^\alpha(L^q)$ , then

$$\int_a^b f(t) D_{a+}^\alpha g(t) dt = \int_a^b g(t) D_{b-}^\alpha f(t) dt.$$

**Lemma 1.3.7.** If  $g(t) \in L^1(-T, 0)$ ,  $T > 0$ ,  $\alpha > 0$ ,  $\lambda \in \mathbb{C}$ , then

$$y(t) - \frac{\lambda}{\Gamma(\alpha)} \int_t^0 (s-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(s-t)^\alpha] y(s) ds = g(t),$$

integral equation has the following unique solution

$$y(t) = g(t) + \lambda \int_t^0 (s-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(s-t)^\alpha] g(s) ds.$$

The solution of the integral equation with the left-sided Riemann-Liouville integral operator second kind was found in [94] by means of the Laplace transform. For ascertaining the result of Lemma 1.3.7 one can easily check by substituting the result into the equation.

**Definition 1.3.8.** [66] The left-sided  ${}^C D_{a+}^\alpha f(t)$  and right-sided  ${}^C D_{b-}^\alpha f(t)$  Caputo derivatives of order  $\alpha$  ( $n - 1 < \alpha \leq n$ ) are defined by

$${}^C D_{a+}^\alpha f(t) = D_{a+}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right]$$

and

$${}^C D_{b-}^\alpha f(t) = D_{b-}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (b-t)^k \right]$$

respectively.

**Lemma 1.3.9.** Let  $\alpha > 0$  and if  $f(t) \in AC^n[a, b]$ , then Caputo fractional derivative exist almost everywhere on  $[a, b]$ . Moreover  ${}^C D_{a+}^\alpha f(t)$  and  ${}^C D_{b-}^\alpha f(t)$  are represented by

$${}^C D_{a+}^\alpha f(t) = I_{a+}^{n-\alpha} D^n f(t)$$

and

$${}^C D_{b-}^\alpha f(t) = (-1)^n I_{b-}^{n-\alpha} D^n f(t)$$

respectively, where  $D = d/dt$ ,  $n = [\alpha] + 1$ .

**Definition 1.3.10.** The right-sided  $D_{a+}^{\alpha, \mu}$  and the left-sided  $D_{b-}^{\alpha, \mu}$  Hilfer fractional derivatives of order  $\alpha$  ( $0 < \alpha \leq 1$ ) and type  $\mu$  ( $0 \leq \mu \leq 1$ ) are defined by

$$D_{a+}^{\alpha, \mu} f(t) = I_{a+}^{\mu(1-\alpha)} \frac{d}{dt} I_{a+}^{(1-\mu)(1-\alpha)} f(t), \quad (1.3.1)$$

and

$$D_{b-}^{\alpha, \mu} f(t) = -I_{b-}^{\mu(1-\alpha)} \frac{d}{dt} I_{b-}^{(1-\mu)(1-\alpha)} f(t). \quad (1.3.2)$$

When  $\mu = 0$  (1.3.1)-(1.3.2) yield the classical left-sided and right-sided Riemann-Liouville fractional differential operators of order  $\alpha$  and for  $\mu = 1$ , these denote the left-sided and right-sided Caputo fractional derivative of order  $\alpha$ .

In [18] M. Bulavatsky introduced the concept of bi-ordinal fractional derivative which is generalization of the Hilfer derivative (1.3.1) in terms of two orders by following form

$$D_{a+}^{\alpha,\mu} f(t) = I_{a+}^{\mu(1-\alpha)} \frac{d}{dt} I_{a+}^{(1-\mu)(1-\beta)} f(t), \quad (0 < \alpha, \beta \leq 1; 0 \leq \mu \leq 1). \quad (1.3.3)$$

### 1.3.2 Representations of the bi-ordinal Hilfer fractional derivative

**Definition 1.3.11.** Let  $I_{a+}^{(1-\mu)(1-\beta)} f(t) \in AC^n[a, b]$  and  $I_{b-}^{(1-\mu)(1-\beta)} f(t) \in AC^n[a, b]$ . Then the left-sided and right-sided bi-ordinal Hilfer fractional derivative of orders  $\alpha$  ( $n-1 < \alpha \leq n$ ) and  $\beta$  ( $n-1 < \beta \leq n$ ) type  $\mu \in [0, 1]$  are defined as follows:

$$D_{a+}^{(\alpha,\beta)\mu} f(t) = I_{a+}^{\mu(n-\alpha)} \left( \frac{d}{dt} \right)^n I_{a+}^{(1-\mu)(n-\beta)} f(t), \quad (1.3.4)$$

$$D_{b-}^{(\alpha,\beta)\mu} f(t) = I_{b-}^{\mu(n-\alpha)} \left( -\frac{d}{dt} \right)^n I_{b-}^{(1-\mu)(n-\beta)} f(t). \quad (1.3.5)$$

**Remark 1.3.12.** We have the following comments:

1) The bi-ordinal Hilfer derivative  $D_{a+}^{(\alpha,\beta)\mu} f(t)$  can be written as

$$\begin{aligned} D_{a+}^{(\alpha,\beta)\mu} f(t) &= I_{a+}^{\mu(n-\alpha)} \left( \frac{d}{dt} \right)^n I_{a+}^{(1-\mu)(n-\beta)} f(t) = I_{a+}^{\mu(n-\alpha)} \left( \frac{d}{dt} \right)^n I_{a+}^{n-\gamma} f(t) \\ &= I_{a+}^{\mu(n-\alpha)} D_{a+}^{\gamma} f(t) = I_{a+}^{\gamma-\delta} D_{a+}^{\gamma} f(t), \end{aligned}$$

for  $t \in [a, b]$ , where  $\gamma = \beta + \mu(n - \beta)$  and  $\delta = \beta + \mu(\alpha - \beta)$ .

2) In general, (1.3.4) is also preserved as (1.4) in terms of its interpolation concept. Specifically, when  $\mu = 0$ , (1.3.4) gives Riemann-Liouville fractional derivative of  $\beta$  order and for  $\mu = 1$ , the bi-ordinal Hilfer fractional derivative (1.3.4) expresses the Caputo fractional derivative of order  $\alpha$  i.e.,

$$D_{a+}^{(\alpha,\beta)\mu} f(t) = \begin{cases} D_{a+}^{\beta} f(t), & \mu = 0, \\ {}^C D_{a+}^{\alpha} f(t), & \mu = 1. \end{cases}$$

3) The parameters  $\gamma$  and  $\delta$  satisfy the following inequalities:

$$n - 1 < \gamma \leq n, \quad \gamma > \beta, \quad n - 1 < \delta \leq n, \quad n \in \mathbb{N}.$$

These comments are also true for  $D_{b-}^{(\alpha, \beta)\mu} f(t)$ .

**Proposition 1.3.13.** *Let  $\mu \in (0, 1)$ . An equivalent form of the bi-ordinal Hilfer derivative for  $0 < \delta \leq 1$  and  $0 < \gamma \leq 1$  can be written as follows*

$$D_{a+}^{(\alpha, \beta)\mu} f(t) = {}^C D_{a+}^{\delta} f(t) + \frac{f(a+)(t-a)^{-\delta}}{\Gamma(1-\delta)}.$$

*Proof.* Let us introduce the following assignation with assuming  $\mu \neq 1$ :

$$\varphi(t) = \frac{d}{dt} I_{a+}^{(1-\mu)(1-\beta)} f(t) = \frac{d}{dt} I_{a+}^{1-\gamma} f(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} f(s) ds,$$

here  $1 - \gamma = (1 - \mu)(1 - \beta)$ .

Integration by parts yields

$$\begin{aligned} \varphi(t) &= \frac{d}{dt} \frac{1}{\Gamma(1-\gamma)} \left[ -\frac{(t-s)^{1-\gamma}}{1-\gamma} f(s) \Big|_a^t + \int_a^t \frac{(t-s)^{1-\gamma}}{1-\gamma} f'(s) ds \right] \\ &= \frac{d}{dt} \frac{1}{\Gamma(1-\gamma)} \left[ \frac{(t-a)^{1-\gamma}}{1-\gamma} f(a+) + \int_a^t \frac{(t-s)^{1-\gamma}}{1-\gamma} f'(s) ds \right] \\ &= \frac{(t-a)^{-\gamma}}{\Gamma(1-\gamma)} f(a+) + \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} f'(s) ds = I_{a+}^{1-\gamma} f'(t) + \frac{f(a+)(t-a)^{-\gamma}}{\Gamma(1-\gamma)}. \end{aligned}$$

According to above assignation and using the properties of the Riemann-Liouville fractional integral, we write the bi-ordinal Hilfer derivative in the following form

$$\begin{aligned} D_{a+}^{(\alpha, \beta)\mu} f(t) &= I_{a+}^{\mu(1-\alpha)} \varphi(t) = I_{a+}^{\mu(1-\alpha)} \left[ I_{a+}^{1-\gamma} f'(t) + \frac{f(a+)(t-a)^{-\gamma}}{\Gamma(1-\gamma)} \right] \\ &= I_{a+}^{\mu-\mu\alpha+1-\gamma} f'(t) + \frac{f(a+)(t-a)^{\mu-\mu\alpha-\gamma}}{\Gamma(1+\mu-\mu\alpha-\gamma)} \end{aligned}$$

$$= I_{a+}^{1-\delta} f'(t) + \frac{f(a+)(t-a)^{-\delta}}{\Gamma(1-\delta)} = {}^C D_{a+}^{\delta} f(t) + \frac{f(a+)(t-a)^{-\delta}}{\Gamma(1-\delta)},$$

here we have used this assignation  $\delta = \mu - \mu\alpha - \gamma = \beta + \mu(\alpha - \beta)$ . The proof is completed.  $\square$

When  $\mu = 1$ , the calculations in the proof of Proposition lose their meaning and the second idea in the Remark 1.3.12 will be valid.

**Remark 1.3.14.** *If  $0 < \alpha, \beta \leq 1$ ,  $0 \leq \mu \leq 1$ ,  $f(t) \in C_{1-\gamma}[a, b]$ , then the left-side bi-ordinal Hilfer fractional derivative of orders  $\alpha, \beta$  and type  $\mu$  can be expressed equivalently as*

$$D_{a+}^{(\alpha, \beta)\mu} f(t) = D_{a+}^{\delta} \left( f(t) - \frac{I_{a+}^{1-\gamma} f(a)}{\Gamma(\gamma)} (t-a)^{\gamma-1} \right),$$

here  $\gamma = \beta + \mu(1 - \beta)$ ,  $\delta = \beta + \mu(\alpha - \beta)$ ,  $I_{a+}^{1-\gamma} f(a) = \lim_{t \rightarrow a+} I_{a+}^{1-\gamma} f(t)$ .

*Proof.*

$$\begin{aligned} D_{a+}^{(\alpha, \beta)\mu} f(t) &= I_{a+}^{\gamma-\delta} \frac{d}{dt} I_{a+}^{1-\gamma} f(t) = \frac{d}{dt} I_{a+}^1 I_{a+}^{\gamma-\delta} \frac{d}{dt} I_{a+}^{1-\gamma} f(t) \\ &= \frac{d}{dt} I_{a+}^{\gamma-\delta} I_{a+}^1 \frac{d}{dt} I_{a+}^{1-\gamma} f(t) = \frac{d}{dt} I_{a+}^{\gamma-\delta} \left[ I_{a+}^{1-\gamma} f(t) - I_{a+}^{1-\gamma} f(a) \right] \\ &= \frac{d}{dt} I_{a+}^{1-\delta} f(t) - \frac{d}{dt} \frac{I_{a+}^{1-\gamma} f(a)}{\Gamma(1+\gamma-\delta)} (t-a)^{\gamma-\delta} \\ &= \frac{d}{dt} I_{a+}^{1-\delta} \left[ f(t) - \frac{I_{a+}^{1-\gamma} f(a)}{\Gamma(1+\gamma-\delta)} I_{a+}^{\delta-1} (t-a)^{\gamma-\delta} \right] \\ &= \frac{d}{dt} I_{a+}^{1-\delta} \left[ f(t) - \frac{I_{a+}^{1-\gamma} f(a)}{\Gamma(\gamma)} (t-a)^{\gamma-1} \right] = D_{a+}^{\delta} \left( f(t) - \frac{I_{a+}^{1-\gamma} f(a)}{\Gamma(\gamma)} (t-a)^{\gamma-1} \right). \end{aligned}$$

The proof is completed.  $\square$

**Lemma 1.3.15.** *If  $I_{0+}^{n-\gamma} g \in AC^n(a, b)$ ,  $n-1 < \alpha, \beta \leq n$  and  $0 \leq \mu \leq 1$ , then*

$$I_{0+}^{\delta} D_{0+}^{(\alpha, \beta)\mu} g(t) = g(t) - \sum_{k=1}^n \frac{t^{\gamma-k}}{\Gamma(\gamma-k+1)} \left[ \lim_{t \rightarrow 0+} \left( \frac{d}{dt} \right)^{n-k} I_{0+}^{n-\gamma} g(t) \right] =$$

$$= g(t) - \sum_{k=1}^n \frac{C_{n-k} \cdot t^{\gamma-k}}{\Gamma(\gamma-k+1)},$$

where  $\delta = \beta + \mu(\alpha - \beta)$ ,  $\gamma = \beta + \mu(n - \beta)$ ,  $C_{n-k} = \lim_{t \rightarrow 0+} \left(\frac{d}{dt}\right)^{n-k} I_{0+}^{n-\gamma} g(t)$ .

*Proof.* The proof of Lemma 1.3.15 can be derived from the Remark 1.3.12 and the composition  $I_{a+}^{\alpha} D_{a+}^{\alpha}$  of the Riemann-Liouville fractional integration  $I_{a+}^{\alpha}$  and differentiation operator  $D_{a+}^{\alpha}$  presented in Property 1.3.5.  $\square$

Similarly, we can represent all comments for the right-sided bi-ordinal Hilfer derivative  $D_{b-}^{(\alpha,\beta)\mu} f$ .

The Laplace transform of the left-sided bi-ordinal Hilfer's fractional derivative can be presented by

$$\begin{aligned} \mathcal{L}\{D_{0+}^{(\alpha,\beta)\mu} f(t)\} &= s^{\beta+\mu(\alpha-\beta)} \mathcal{L}\{f(t)\} - \\ &- s^{\mu(\alpha-n)} \sum_{k=0}^{n-1} s^{n-k-1} \left(\frac{d}{dt}\right)^k I_{0+}^{(1-\mu)(n-\beta)} f(0+), \end{aligned} \quad (1.3.6)$$

where the Laplace transform of a function  $f(t)$  is defined by

$$\mathcal{L}\{f\}(s) := \int_0^{\infty} e^{-st} f(t) dt.$$

Now let us consider the following problem intended of finding the solution of the following equation

$$a D_t^{(\alpha,\beta)\mu} u(t) + b u(t) = f(t), \quad (1.3.7)$$

which satisfies the weighted initial condition

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} u(t) = u_0. \quad (1.3.8)$$

Where  $f(t)$  is a given source function  $0 < \alpha, \beta \leq 1$ ,  $0 \leq \mu \leq 1$  and  $a, b, u_0 \in \mathbb{R}$ .

This type of problem was considered by R.Hilfer [48] for the first time and in [49] authors developed it using the operational method for solving the

general equation with higher order and presented the spaces of the solution and given functions. The following lemma is an analogy of the result given in [49], [18].

**Lemma 1.3.16.** *If  $f \in C_{1-\gamma}[0, T]$ , then the solution  $u \in C_{1-\gamma}[0, T]$  of (1.3.7)-(1.3.8) can be found uniquely as follows*

$$u(t) = u_0 t^{\gamma-1} E_{\delta, \gamma} \left( -\frac{b}{a} t^\delta \right) + \frac{1}{a} \int_0^t (t-s)^{\delta-1} E_{\delta, \delta} \left[ -\frac{b}{a} (t-s)^\delta \right] f(s) ds, \quad (1.3.9)$$

where  $\delta = \beta + \mu(\alpha - \beta)$ ,  $\gamma = \beta + \mu(1 - \beta)$ .

### 1.3.3 Regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator

**Definition 1.3.17.** *The Erdelyi-Kober (E-K) fractional integral of a function  $f(t) \in C_\mu$   $\mu \geq -\beta(\gamma + 1)$  with arbitrary parameters  $\delta > 0, \gamma \in \mathbb{R}$  and  $\beta > 0$  is defined as ([68])*

$$I_{\beta; a+}^{\gamma, \delta} f(t) = \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_a^t (t^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} f(\tau) d(\tau^\beta),$$

which can be reduced up to a weight to  $I_{a+}^q f(t)$  (Riemann-Liouville fractional integral) at  $\gamma = 0$  and  $\beta = 1$ , and Erdelyi-Kober fractional derivative of  $f(t) \in C_\mu^{(n)}$  for  $n-1 < \delta \leq n, n \in \mathbb{N}$  is defined by

$$D_{\beta, a+}^{\gamma, \delta} f(t) = \prod_{j=1}^n \left( \gamma + j + \frac{t}{\beta} \frac{d}{dt} \right) (I_{\beta, a+}^{\gamma+\delta, n-\delta} f(t)),$$

where  $C_\mu^{(n)}$  is the weighted space of continuous functions defined in 1.2.1.

In 1965, by Soviet mathematicians, Ditkin and Prudnikov, the following operator was considered and investigated in [25] (as a collection of papers published since 1962)

$$D_L := \frac{d}{dt} t \frac{d}{dt}. \quad (1.3.10)$$

For the information, we refer that, [22] Dattolli and Ricci, also considered the so-called Laguerre derivative operator later in 2003 as,

$$D_{nL} := \underbrace{\frac{d}{dt} t \dots \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt}}_{n+1 \text{ derivatives}}, \quad (1.3.11)$$

which (1.3.10) can be determined as a particular case when  $n = 1$ .

However, I. Dimovski [23], earlier in 1966, introduced *the hyper-Bessel differential operator* of higher (integer) order  $m \geq 1$

$$B := t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} t^{\alpha_2} \dots \frac{d}{dt} t^{\alpha_{m-1}} \frac{d}{dt} t^{\alpha_m}, \quad t > 0 \quad (1.3.12)$$

with  $\beta = m - (\alpha_0 + \alpha_1 + \dots + \alpha_m) > 0$  and Laguerre differential operator (1.3.11) is only a special case of the so-called hyper-Bessel differential operators (1.3.12). This can be seen as a generalization of Bessel differential operator  $B_\nu$  of order  $m = 2$  when  $\alpha_0 = \nu' - 1$ ,  $\alpha_1 = -2\nu' + 1$ ,  $\alpha_2 = \nu'$ ;  $\nu' = \pm\nu$ .

Dimoski developed operational calculus for the corresponding linear right inverse operator  $L$  of (1.3.12), such that  $BL = I$ , where  $L$  is a hyper-Bessel integral operator,  $I$  is identity operator. Later, by using operational calculus he also proposed fractional powers  $B^\theta$  as a convolution products [24]. Using these fractional powers, further represented by integral operators of Riemann-Liouville and Erdelyi-Kober type, Kiryakova introduced the theory of the operators of the generalized fractional calculus. For more details on the theory of the hyper-Bessel operators and its applications, see Kiryakova [68], Ch.3 and references therein.

Moreover, in [68], as well as, Kiyakova presented fractional multi-order analogues of the hyper-Bessel operators (1.3.12) identified as particular cases of the generalized fractional derivatives of multi-order  $\delta = (\delta_1, \dots, \delta_m)$

$$\mathcal{D} = t^{\alpha_0} \left( \frac{d}{dt} \right)^{\delta_1} t^{\alpha_1} \left( \frac{d}{dt} \right)^{\delta_2} t^{\alpha_2} \dots \left( \frac{d}{dt} \right)^{\delta_{m-1}} t^{\alpha_{m-1}} \left( \frac{d}{dt} \right)^{\delta_m} t^{\alpha_m}$$



In 2013, the fractional order version of (1.3.11) was analyzed by Garra and Polito in [38]

$$\underbrace{\frac{d^\nu}{dt^\nu} t^\nu \dots \frac{d^\nu}{dt^\nu} t^\nu \frac{d^\nu}{dt^\nu} t^\nu \frac{d^\nu}{dt^\nu} t^\nu}_{n+1 \text{ derivatives}}$$

as a hyper-Bessel type operator, where  $\frac{d^\nu}{dt^\nu} t^\nu$  stands for the Caputo fractional derivative.

In [39], Roberto Garra et.al. considered a particular operator that is suitable to generalize the standard process of relaxation considering both memory effects of power law type and time variability of the characteristic coefficient. They applied the McBride-Lamb theory of the fractional powers of Bessel-type operators and as a result, they obtained an explicit representation of the fractional order operator in terms of Erdelyi-Kober and Hadamard integrals.

**Definition 1.3.18.** *A particular hyper-Bessel operator of order  $0 < \alpha \leq 1$  identified in terms of E-K integral or derivative can be represented as follows*

$$\left(t^\theta \frac{d}{dt}\right)^\alpha f(t) = \begin{cases} (1-\theta)^\alpha t^{-\alpha(1-\theta)} I_{1-\theta}^{0,-\alpha} f(t), & \text{if } \theta < 1, \\ (\theta-1)^\alpha I_{1-\theta}^{-1,-\alpha} t^{(1-\theta)\alpha} f(t), & \text{if } \theta > 1. \end{cases} \quad (1.3.13)$$

We note that  $I_\beta^{\gamma,-\alpha} := D_\beta^{\gamma-\alpha,\alpha}$  is the interpretation of E-K integral for negative order such that

$$I_\beta^{\gamma,-\alpha} := (\gamma - \alpha + 1) I_\beta^{\gamma,1-\alpha} f(t) + \frac{1}{\beta} I_\beta^{\gamma,1-\alpha} \left(t \frac{d}{dt} f(t)\right).$$

The operator (1.3.13) coincides with the Riemann-Liouville fractional derivative when  $\theta = 0$ .

When  $\theta = 1$  the explicit form of the operator  $\left(t \frac{d}{dt}\right)^\alpha$  comes from the theory of fractional powers of operators such that

$$\left(t \frac{d}{dt}\right)^\alpha f(t) = \delta \left(\mathcal{J}_{t_0^+}^{1-\alpha} f\right)(t), \quad 0 < \alpha < 1, \quad (1.3.14)$$

where  $\delta = t \frac{d}{dt}$  and

$$\left( \mathcal{J}_{t_0^+}^{1-\alpha} f \right) (t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \left( \ln \frac{x}{u} \right)^{-\alpha} f(u) \frac{du}{u}, \quad t_0 \geq 0$$

is the Hadamard fractional integral of order  $1 - \alpha$ .

**Definition 1.3.19.** *A regularized Caputo-like counterpart of the operator (1.3.13) is defined for  $\theta < 1$  in terms of E-K fractional order operator such that*

$${}^C \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = (1-\theta)^\alpha t^{-\alpha(1-\theta)} I_{1-\theta}^{0,-\alpha} [f(t) - f(0)] \quad (1.3.15)$$

*or in terms of the hyper-Bessel differential operator*

$${}^C \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) - \frac{f(0)t^{-\alpha(1-\theta)}}{(1-\theta)^{-\alpha}\Gamma(1-\alpha)}, \quad (1.3.16)$$

*where  $f(0)$  is the initial condition.*

Considering above notations a regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator can be written briefly for  $\theta < 1$  and  $\alpha \in (0, 1)$  as

$${}^C \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = p^\alpha t^{-p\alpha} D_p^{-\alpha, \alpha} f(t) - \frac{f(0)p^\alpha t^{-p\alpha}}{\Gamma(1-\alpha)}, \quad (1.3.17)$$

where  $p = 1 - \theta$ .

**Theorem 1.3.20.** *[5] If  $f(t) \in C_\mu[0, \infty)$ , then the non-homogeneous fractional differential equation*

$${}^C \left( t^\theta \frac{d}{dt} \right)^\alpha u(t) = -\lambda u(t) + f(t), \quad \alpha \in (0, 1), \quad \theta < 1 \quad (1.3.18)$$

*satisfying the initial condition  $u(0) = u_0$  for  $t > 0$ , has a unique solution*

represented by

$$\begin{aligned}
 u(t) = & u_0 E_{\alpha,1}(\lambda^* t^{p\alpha}) + \frac{1}{p^\alpha \Gamma(\alpha)} \int_0^t (t^p - x^p)^{\alpha-1} f(x) d(x^p) + \\
 & + \frac{\lambda^*}{p^\alpha} \int_0^t (t^p - x^p)^{2\alpha-1} E_{\alpha,2\alpha}[\lambda^* (t^p - x^p)^\alpha] f(x) d(x^p),
 \end{aligned} \tag{1.3.19}$$

where  $p = 1 - \theta$  and  $\lambda^* = -\frac{\lambda}{p^\alpha}$ .

**Remark 1.3.21.** The solution (1.3.19) also can be written as follows

$$\begin{aligned}
 u(t) = & u_0 E_{\alpha,1}(\lambda^* t^{p\alpha}) + \\
 & + \frac{1}{p^\alpha} \int_0^t (t^p - x^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^* (t^p - x^p)^\alpha] f(x) d(x^p).
 \end{aligned} \tag{1.3.20}$$

## 1.4 Important properties of the Mittag-Leffler function

Since the introduction of the Mittag-Leffler function by Swedish Mathematician Magnus Gustaf Mittag-Leffler in connection with methods for summation of divergent series, many applications of this function or other functions in this type have been revealed and investigated. The two parameter Mittag-Leffler (M-L) function is an entire function and given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R}.$$

In the sequel lemmas, we present an important properties of Mittag-Leffler type functions.

**Lemma 1.4.1.** (see [85]) Let  $\alpha < 2$ ,  $\beta \in \mathbb{R}$  and  $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ . Then the following estimate holds true

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1 + |z|}, \quad \mu \leq |\arg z| \leq \pi, \quad |z| \geq 0.$$

Here and in the rest of the paper,  $M$  is a positive constant.

**Lemma 1.4.2.** [17] For every  $\alpha \in (0, 1]$ ,  $\beta > \alpha$  and  $x \geq 0$  one has

$$\frac{1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}x} \leq \Gamma(\beta)E_{\alpha,\beta}(-x) \leq \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}x}.$$

The Laplace transform of the Mittag-Leffler function is given in the following lemma.

**Lemma 1.4.3.** [66] For any  $\alpha > 0$ ,  $\beta > 0$  and  $\lambda \in \mathbb{C}$ , we have

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \quad (\operatorname{Re}(s) > |\lambda|^{1/\alpha}).$$

**Lemma 1.4.4.** [66] If  $\alpha > 0$  and  $\beta \in \mathbb{C}$ , then the following recurrence formula holds:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z).$$

We also remind that the fractional integration of the Mittag-Leffler functions which will be used in the sequel:

$$\frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} E_{\alpha,\beta}(\lambda z^\alpha) z^{\beta-1} dz = t^{\beta+\gamma-1} E_{\alpha,\beta+\gamma}(\lambda t^\alpha), \quad \beta > 0, \gamma > 0. \quad (1.4.1)$$

In 1969, the Mittag-Leffler function was generalized by Indian mathematician Tilak Raj Prabhakar called three parameter Mittag-Leffler function or Prabhakar function as follows [90]

$$E_{\alpha,\beta}^\gamma(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\alpha n + \beta)} z^n, \quad \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\alpha) > 0, \gamma > 0,$$

where  $(\gamma)_n = \gamma(\gamma+1)\dots(\gamma+n-1)$  is Pochhammer's symbol. When  $\gamma = 1$ , Prabhakar function becomes into two parameter Mittag-Leffler function  $E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z)$  and it is possible to write the Prabhakar function, when  $\gamma = 2, 3$ , in terms of the two-parameter Mittag-Leffler function as follows:

$$E_{\alpha,\beta}^2(z) = \frac{1}{\alpha} \left[ E_{\alpha,\beta-1}(z) + (1 - \beta + \alpha) E_{\alpha,\beta}(z) \right],$$

$$E_{\alpha,\beta}^3(z) = \frac{1}{2\alpha^2} \left[ E_{\alpha,\beta-2}(z) + (1-\alpha)E_{\alpha,\beta-1}(z) + (1-\beta+2\alpha)(1-\beta+\alpha)E_{\alpha,\beta}(z) \right].$$

Later, we use the properties of a Wright-type function studied by A. Pskhu [86], defined as

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}, \quad \alpha > 0, \quad \alpha > \beta.$$

Mittag-Leffler function can be determined by Wright-type function as a special case  $E_{\alpha,\beta}(z) = e_{\alpha,0}^{\beta,1}(z)$ . So, we can record some results of Mittag-Leffler function which can be reduced from the properties of Wright-type function.

**Lemma 1.4.5.** [86] *If  $\pi \geq |\arg z| > \frac{\pi\alpha}{2} + \varepsilon$ ,  $\varepsilon > 0$ , then the following relations are valid for  $z \rightarrow \infty$ :*

$$\begin{aligned} \lim_{|z| \rightarrow \infty} E_{\alpha,\beta}(z) &= 0, \\ \lim_{|z| \rightarrow \infty} z E_{\alpha,\beta}(z) &= -\frac{1}{\Gamma(\beta - \alpha)}. \end{aligned}$$

## Chapter 2

# Solvability of boundary value problems for mixed equations involving hyper-Bessel fractional differential operator

Studying mixed-type equations is one of the well-developed parts of the contemporary theory of partial differential equations (PDEs). Especially, mixed PDEs play an important role in the theory of transonic flow, and they arise in particular boundary value problems known as the Tricomi and Frankl problems. Origins of these researches date back to the beginning of the last century when the Tricomi introduced the following equation as it is of the elliptic and hyperbolic type according to sign of the coefficient  $y$ :

$$yu_{xx} + u_{yy} = 0 \tag{2.0.1}$$

The Tricomi's research [111] gave a rise to the theory of PDEs of mixed type with certain boundary conditions. Transonic flows involve a transition from the subsonic to the supersonic region via the sonic curve. As a result, transonic flows are very interesting phenomena that appear in aerodynamics and hydrodynamics.

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However, we must admit that the most fundamental results on mixed type equations with their applications appeared in Chaplygin's work [20] in 1904. He stated that the study of a mixed type equation known as the Chaplygin equation is intimately related to the science of gas flow which is more general than Tricomi's equation. This equation describes the two-dimensional adiabatic potential flow of a perfect gas as well.

Gellerstedt generalized [43] the Tricomi problem in 1935 by substituting a power of  $y$  for the coefficient  $y$  in the equation (2.0.1) i.e,

$$\operatorname{sgn}(y)|y|^m u_{xx} + u_{yy} = 0, \quad m > 0. \quad (2.0.2)$$

In 1945, Frankl investigated the Tricomi and Chaplygin problems which is related to the study of gas flow with nearly sonic speeds [34].

Some applications of boundary-value problems for mixed parabolic-hyperbolic type equations appeared in practical problems [114], [42], [31]. Later, many kind of problems, including nonlocal boundary value problems were studied by many authors (see, for instance monographs [27], [81]). Interesting application of a boundary value problems with integral conjugation condition for mixed type equations was also noticed in [102]. Different studies devoted to the boundary value problems for mixed elliptic-hyperbolic [82] and hyperbolic [26] type equations were published. Several local and nonlocal boundary problems with integral conjugation conditions for parabolic-hyperbolic type equations with one or two lines of type changing were studied in [61]. In these problems, form of conjugation condition depends on equation in a hyperbolic part of considered mixed type equations.

In next works, authors investigated more general equations, domains and conjugation conditions (not depending on certain form of equation). For instance, fractional diffusion equation and wave equation were widely investigated in different domains in [60], [2], [64], [62], [28], [51], [13]. We note recent work

[59], where the Tricomi type problem with integral conjugation conditions for a mixed type (PDEs) with sub-diffusion equation involving Hilfer fractional differential operator has been studied. Frankl-type problems for mixed hyperbolic-parabolic type equations were studied in [71], [53], [87] which are considered a good example for non-local problems.

In the following sections we investigate analogues of the Tricomi and Frankl type problems for mixed equations.

This chapter is based on the articles [119] and [120], which have been already published in *Bulletin of the Institute of Mathematics* and *Montes Taurus Journal of Pure and Applied Mathematics*, respectively.

## 2.1 An analogue of the Tricomi problem for a mixed type equation with the hyper-Bessel fractional differential operator

In this section, we are interested in studying boundary value problem with integral form conjugation condition in a mixed domain consisted of characteristic triangle and rectangle, for a mixed type partial differential equation with the regularized Caputo-like counterpart of hyper-Bessel fractional differential operator.

### Formulation of a problem

In a domain  $\Omega = \Omega_1 \cup \Omega_2 \cup AB$  let us consider the following mixed type equation

$$0 = \begin{cases} {}^C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t), & t > 0 \\ u_{tt}(x, t) - u_{xx}(x, t), & t < 0 \end{cases} \quad (2.1.1)$$



where  $\alpha, \theta, T$  are real numbers such that  $0 < \alpha < 1, \theta < 1, T > 0$  conditions,

$$\Omega_1 = \{(x, t) : 0 < x < 1, 0 < t < T\}, \quad AB = \{(x, t) : t = 0, 0 < x < 1\},$$

$$\Omega_2 = \left\{ (x, t) : -t < x < t + 1, -\frac{1}{2} < t < 0 \right\},$$

${}^C(t^\theta \frac{\partial}{\partial t})^\alpha f(t)$  stands for the regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator of order  $\alpha$  ( $0 < \alpha < 1$ ) defined in Definition 1.3.19.

**Problem T.** To find a function  $u(x, t)$  which is continuous in  $\bar{\Omega}$ , its hyper-Bessel derivative is continuous in  $\Omega_1$  and it has continuous second order partial derivatives in  $\Omega_2$ , and it satisfies Eq.(2.1.1) in  $\Omega$  together with boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \quad (2.1.2)$$

$$u(x/2, -x/2) = \psi(x), \quad 0 \leq x \leq 1, \quad (2.1.3)$$

conjugation condition on  $AB$

$$\lim_{t \rightarrow +0} t^{1-(1-\theta)\alpha} u_t(x, t) = \gamma_1 u_t(x, -0) +$$

$$+ \gamma_2 \int_0^1 u_t(z, -0) P(x, z) dz, \quad 0 < x < 1, \quad (2.1.4)$$

where  $\gamma_1, \gamma_2 \in \mathbb{R}$ ,  $P(x, t)$  is given function.

The following statements provide the unique solvability of the Problem T.

**Theorem 2.1.1.** *If  $\gamma_2 \geq 0$ ,*

$$\frac{\partial}{\partial z} P(x, z) = P_1(x) P_1(z), \quad (2.1.5)$$

*are fulfilled, then Problem T has a unique solution.*

**Theorem 2.1.2.** *Let all conditions of the Theorem 2.1.1 be valid. If  $\psi(x) \in C^1[0, 1]$  and  $P(x, z)$  has continuous partial derivatives in  $[0, 1] \times [0, 1]$ , then there exist a unique solution of the Problem T represented as*

$$\begin{aligned}
 u(x, t) = & 2\Theta(t) \int_0^1 \left[ F(\xi) + \int_0^1 F(z) R(\xi, z) dz \right] \times \\
 & \times \sum_{k=1}^{\infty} E_{\alpha, 1} \left( -\frac{(k\pi)^2}{(1-\theta)^2} t^{(1-\theta)\alpha} \right) \sin(k\pi\xi) \sin(k\pi x) d\xi + \\
 & + \frac{\Theta(-t)}{2} \left\{ F(x+t) + F(x-t) \int_0^1 F(\xi) [R(x+t, \xi) + R(x-t, \xi)] d\xi + \right. \\
 & \left. + \int_{x-t}^{x+t} \left[ F''(z) + \int_0^1 F(\xi) \frac{\partial^2 R(z, \xi)}{\partial z^2} d\xi - 2\psi'(z) \right], dz \right\} \quad (2.1.6)
 \end{aligned}$$

where  $A = \gamma_1 \Gamma(\alpha) (1-\theta)^\alpha$ ,  $\Theta(t) = 1$  for  $t > 0$  and  $\Theta(t) = 0$  for  $t < 0$ ,  $R(x, \xi)$  is the resolvent-kernel of

$$K(x, \xi) = -A\gamma_2 \int_0^1 G_0(x, \xi) \frac{\partial P(\xi, z)}{\partial z} dz, \quad (2.1.7)$$

$$\begin{aligned}
 F(x) = & -2A\gamma_1 \int_0^1 G_0(x, \xi) \psi'(\xi) d\xi \\
 & - 2A\gamma_2 \int_0^1 G_0(x, \xi) d\xi \int_0^1 \psi'(\xi) P(\xi, z) dz, \quad (2.1.8)
 \end{aligned}$$

$$G_0(x, \xi) = \frac{1}{A[e^{Ax} - e^{A(x-1)}]} \begin{cases} (1 - e^{A\xi}) (1 - e^{A(x-1)}), & 0 \leq \xi \leq x, \\ (1 - e^{Ax}) (1 - e^{A(\xi-1)}), & x \leq \xi \leq 1. \end{cases} \quad (2.1.9)$$

**Proof of existence result.** First, we prove Theorem 2.1.2.

Let us introduce a notation

$$\tau^+(x) = u(x, +0), \quad 0 \leq x \leq 1. \quad (2.1.10)$$

Solution of the Eq.(2.1.1) in  $\Omega_1$  which satisfies conditions (2.1.2), (2.1.10) can be written as follows [5]:

$$u(x, t) = 2 \int_0^1 \tau^+(\xi) \sum_{k=1}^{\infty} E_{\alpha,1} \left[ -\frac{(k\pi)^2}{(1-\theta)^2} t^{(1-\theta)\alpha} \right] \times \quad (2.1.11)$$

$$\times \sin(k\pi\xi) \sin(k\pi x) d\xi.$$

Using representation (2.1.11), we evaluate  $t^{1-(1-\theta)\alpha} u_t(x, t)$ :

$$t^{1-(1-\theta)\alpha} u_t(x, t) = \sum_{k=1}^{\infty} \left( -\frac{(k\pi)^2}{(1-\theta)^\alpha} \right) \tau_k^+ \cdot E_{\alpha,\alpha} \left( -\frac{(k\pi)^2}{(1-\theta)^\alpha} t^{(1-\theta)\alpha} \right) \sin k\pi x,$$

where  $\tau_k^+ = 2 \int_0^1 \tau^+(\xi) \sin k\pi\xi d\xi$ .

We introduce another notation, namely

$$\nu^+(x) = \lim_{t \rightarrow +0} t^{1-(1-\theta)\alpha} u_t(x, t), \quad 0 < x < 1. \quad (2.1.12)$$

Considering above-given evaluations, from (2.1.11) we obtain the following functional relation on  $AB$  deduced from  $\Omega_1$  as  $t \rightarrow +0$ :

$$\nu^+(x) = \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha)} \tau^{+\prime\prime}(x), \quad 0 < x < 1. \quad (2.1.13)$$

Here we have used  $2 \int_0^1 \tau^+(\xi) \sin k\pi\xi d\xi = -\frac{2}{(k\pi)^2} \int_0^1 \tau^{+\prime\prime}(\xi) \sin k\pi\xi d\xi$ , which is true due to  $\tau^+(0) = \tau^+(1) = 0$  (see condition (2.1.2) and notation (2.1.10)).

Now we will establish another functional relation on  $AB$  which will be reduced from  $\Omega_2$ . For this aim, we use a solution of the following Cauchy problem:

$$\begin{cases} u_{xx} - u_{tt} = 0, \\ u(x, -0) = \tau^-(x), \quad 0 < x < 1, \\ u_t(x, -0) = \nu^-(x), \quad 0 < x < 1, \end{cases}$$

which has a form [107]

$$u(x, t) = \frac{1}{2} \left[ \tau^-(x+t) + \tau^-(x-t) + \int_{x-t}^{x+t} \nu^-(z) dz \right]. \quad (2.1.14)$$

We substitute (2.1.14) into (2.1.3) and deduce

$$\nu^-(x) = \tau^{-'}(x) - 2\psi'(x), \quad 0 < x < 1. \quad (2.1.15)$$

Considering conjugation condition (2.1.4), from functional relations (2.1.12) and (2.1.15) we get

$$\tau^{+''}(x) - A\tau^{+'}(x) = F_0(x), \quad (2.1.16)$$

where  $A = \gamma_1 \Gamma(\alpha) (1 - \theta)^\alpha$ ,

$$F_0(x) = \Gamma(\alpha) (1 - \theta)^\alpha \left[ \gamma_2 \int_0^1 \tau^{+'}(z) P(x, z) dz - \right. \\ \left. - 2\psi'(x) - 2\gamma_2 \int_0^1 \psi'(z) P(x, z) dz \right]. \quad (2.1.17)$$

Boundary conditions (2.1.2) yield

$$\tau^+(0) = 0, \quad \tau^+(1) = 0. \quad (2.1.18)$$

Solution of (2.1.16), (2.1.18) can be written as [27]

$$\tau^+(x) = \int_0^1 G_0(x, \xi) F_0(\xi) d\xi, \quad (2.1.19)$$

where

$$G_0(x, \xi) = \frac{1}{A [e^{Ax} - e^{A(x-1)}]} \begin{cases} (1 - e^{A\xi}) (1 - e^{A(x-1)}), & 0 \leq \xi \leq x, \\ (1 - e^{Ax}) (1 - e^{A(\xi-1)}), & x \leq \xi \leq 1. \end{cases}$$

Substituting (2.1.17) into (2.1.19), after integration by parts, we will get

$$\tau^+(x) - \int_0^1 \tau^+(\xi) K(x, \xi) d\xi = F(x), \quad 0 \leq x \leq 1, \quad (2.1.20)$$

where  $K(x, \xi)$ ,  $F(x)$  are defined by formulas (2.1.7), (2.1.8), respectively. Due to (2.1.9)  $K(x, \xi)$  is continuous and if  $F(x)$  is continuously differentiable (for this we assume that  $\psi(x) \in C^1[0, 1]$ ,  $P(x, z) \in C^1([0, 1] \times [0, 1])$ ), hence the solution of the second kind Fredholm integral equation (2.1.20) can be represented via resolvent-kernel [112]:

$$\tau^+(x) = F(x) + \int_0^1 F(\xi) R(x, \xi) d\xi, \quad (2.1.21)$$

where  $R(x, \xi)$  is a resolvent-kernel of  $K(x, \xi)$ .

This will complete the proof of the Theorem 2.1.2. □

**Proof of uniqueness of the result.** Now we prove Theorem 2.1.1, which states a uniqueness of the obtained solution.

For this aim, we multiply equality (2.1.13) by  $\tau^+(x)$  and integrate along  $AB$ :

$$\int_0^1 \tau^+(x) \nu^+(x) dx = \frac{1}{\Gamma(\alpha)(1-\theta)^\alpha} \int_0^1 \tau^+(x) \tau^{+''}(x) dx.$$

Considering  $\int_0^1 \tau^+(x) \tau^{+''}(x) dx = - \int_0^1 [\tau^{+'}(x)]^2 dx$ , we deduce

$$\int_0^1 \tau^+(x) \nu^+(x) dx + \frac{1}{\Gamma(\alpha)(1-\theta)^\alpha} \int_0^1 [\tau^{+'}(x)]^2 dx = 0. \quad (2.1.22)$$

Let us first consider the following integral

$$I = \int_0^1 \tau^+(x) \nu^+(x) dx. \quad (2.1.23)$$

Considering (2.1.4) and (2.1.15) at  $\psi(x) \equiv 0$ , after integration by parts we get

$$I = \gamma_2 \int_0^1 \tau^+(x) dx \int_0^1 \tau^+(z) \frac{\partial}{\partial z} P(x, z) dz. \quad (2.1.24)$$

Suppose that condition (2.1.5) is valid, then from (2.1.24) it follows that

$$I = \gamma_2 \left( \int_0^1 \tau^+(x) P_1(x) dx \right)^2. \quad (2.1.25)$$

If we suppose that  $\gamma_2 \geq 0$ , from (2.1.25) we will get  $I \geq 0$ .

Since  $\Gamma(\alpha) > 0$  for all  $\alpha > 0$ , and  $(1 - \theta)^\alpha > 0$  is true for all  $\theta < 1$ , then from (2.1.22) we will have  $\tau^+(x) \equiv 0$ . Further, considering solution of the first BVP for Eq.(2.1.1) in  $\Omega_1$ , we will get  $u(x, t) \equiv 0$  in  $\bar{\Omega}_1$ . Due to the continuity of  $u(x, t)$  in  $\bar{\Omega}$  and (2.1.13)-(2.1.14), one can easily deduce that  $u(x, t) \equiv 0$  in  $\bar{\Omega}$ .

Finally, based on (2.1.11), (2.1.13)-(2.1.14) and (2.1.21), we can represent the solution of the problem by formula (2.1.6).

Theorem 2.1.1 is proved. □

**Remark 2.1.3.** We note that set of functions  $P(x, z)$ , satisfying condition (2.1.5) is not empty. For instance,  $P(x, z) = \exp(x + z)$  or  $P(x, z) = -\sin x \cos z$ .

## 2.2 Frankl-type problem for a mixed equation associated with the hyper-Bessel fractional differential operator

In this section, we are concerned with studying unique solvability of a Frankl-type problem for partial differential equation of mixed PDE involving sub-diffusion equation with the hyper-Bessel fractional derivative and the wave equation. We have used the method of energy integrals for proving uniqueness

of the solution and also method of integral equations for showing the existence of the solution of the problem. We have to note that obtained result can be used in mathematical models of the gas movement in a porous medium. The usage of special fractional derivative can be justified by memory effect. For details we refer E.Karimov's DSc thesis [55].

Let  $\Omega = \Omega_1 \cup \Omega_2 \cup AB$  be a simple-connected domain, where  
 $AB = \{(x, t) : t = 0, 0 < x < 1\}$ ,  $\Omega_1 = \{(x, t) : 0 < t < 1, 0 < x < 1\}$ ,  
 $\Omega_2 = \{(x, t) : -t < x < t + 1, -1/2 < t < 0\}$ .

**Problem F.** To find a function  $u(x, t)$ , which is

- 1)  $u(x, t) \in C(\overline{\Omega}) \cap C^1(\overline{\Omega}_2) \cap C^2(\Omega_2)$ ,  ${}^C(t^\theta \frac{\partial}{\partial t})^\alpha u \in C(\Omega_1)$ ,  $u_{xx} \in C(\Omega_1)$ ;
- 2) satisfies equation

$$0 = \begin{cases} u_{xx}(x, t) - {}^C(t^\theta \frac{\partial}{\partial t})^\alpha u(x, t), & (x, t) \in \Omega_1, \\ u_{xx}(x, t) - u_{tt}(x, t), & (x, t) \in \Omega_2; \end{cases} \quad (2.2.1)$$

- 3) satisfies non-local conditions

$$a_1(z)u(0, z) + b_1(z)u(z/2, -z/2) = c_1(z), \quad 0 \leq z \leq 1, \quad (2.2.2)$$

$$a_2(z)u(1, z) + b_2(z)u((z+1)/2, (z-1)/2) = c_2(z), \quad 0 \leq z \leq 1, \quad (2.2.3)$$

$$a_3(z)u(0, z) + b_3(z)u(1, z) = c_3(z), \quad 0 \leq z \leq 1; \quad (2.2.4)$$

- 4)  $u(x, t)$  satisfies the following conjugating condition

$$\lim_{t \rightarrow +0} t^{1-\alpha(1-\theta)} u_t(x, t) = \lim_{t \rightarrow -0} u_t(x, t). \quad (2.2.5)$$

Here  $0 < \alpha < 1$ ,  $\theta < 1$ ,  $a_i(z)$ ,  $b_i(z)$ ,  $c_i(z)$  ( $i = \overline{1, 3}$ ) are given continuous functions such that

$$\begin{aligned} a_1^2(z) + a_2^2(z) &\neq 0, \quad a_1^2(z) + a_3^2(z) \neq 0, \quad b_1^2(z) + b_2^2(z) \neq 0, \\ b_1^2(z) + b_3^2(z) &\neq 0, \quad a_j^2(z) + b_j^2(z) + c_j^2(z) \neq 0, \quad j = 1, 2, \end{aligned}$$

It is known that Frankl problem [35] has a specific nonlocal condition which connects parts of boundary of the mixed domain.

The present investigation's distinguishing feature is the utilization of non-local Frankl-type conditions. As a result of these nonlocal conditions, which correlate values of the sought function on different parts of the boundary of the considered mixed domain, numerous well phrased local and nonlocal problems can be obtained, which can be analyzed using the same method. On the line of type-changing, using the integral form gluing condition has both a mathematical and an applied aspect. We must keep in mind that certain conjugating conditions have an impact on the solvability of the problem in question; for example, the problem could be reduced to the Fredholm or Volterra integral equations by using the proper conjugating condition on the type-changing line.

For the investigation, we used well-known approaches, although with a few tweaks. Finding the key functional relations on the type-changing line, in particular, necessitates a certain order of stages as well as an analogous reduction of the problem to the somewhat different second kind Fredholm integral equation.

The main result we formulate as the following statement:

**Theorem 2.2.1.** *If  $d_2'(z) > 0$ ,  $d_2 \neq 1$  and  $a_i(z), b_i(z), c_i(z) \in C(0, 1) \cap C^1[0, 1]$ , such that  $a_i(z) \neq 0$ ,  $b_i(z) \neq 0$ ,  $z \in [0, 1]$ ,  $a_1(0) \neq -b_1(0)$ , then a unique solution of the Problem F exists, where*

$$d_2(z) = \frac{a_1(z) b_2(z) b_3(z)}{a_2(z) a_3(z) b_1(z)}.$$

*Proof.* First, we prove the uniqueness of the solution.

### The uniqueness of a solution.

We introduce the following notations and assumptions:

$$u(x, \pm 0) = \tau(x), \quad x \in [0, 1], \quad \tau(x) \in C[0, 1] \cap C^2(0, 1), \quad (2.2.6)$$



$$\lim_{t \rightarrow -0} u_t(x, t) = \nu_2(x), \quad x \in (0, 1), \quad \nu_2(x) \in C^1(0, 1) \cap L^1(0, 1). \quad (2.2.7)$$

Solutions of the Cauchy problem for the equation (2.2.1) in  $\Omega_2$  can be represented by the D'Alembert's formula

$$u(x, t) = \frac{1}{2} \left[ \tau(x+t) + \tau(x-t) + \int_{x-t}^{x+t} \nu_2(z) dz \right]. \quad (2.2.8)$$

We assume that  $a_i, b_i \neq 0$  ( $i = \overline{1, 3}$ ) and from (2.2.2) and (2.2.3), we find that

$$u(0, z) = \frac{c_1(z) - b_1(z) \psi(z)}{a_1(z)} = g_1(z), \quad (2.2.9)$$

$$u(1, z) = \frac{c_2(z) - b_2(z) \varphi(z)}{a_2(z)} = g_2(z), \quad (2.2.10)$$

where

$$u\left(\frac{z+1}{2}, \frac{z-1}{2}\right) = \varphi(z), \quad 0 \leq z \leq 1, \quad (2.2.11)$$

$$u\left(\frac{z}{2}, \frac{-z}{2}\right) = \psi(z), \quad 0 \leq z \leq 1. \quad (2.2.12)$$

We note that functions  $g_1(z), g_2(z), \psi(z), \varphi(z)$  are not known yet.

By substituting the solution (2.2.8) into the condition (2.2.12) and differentiating once, we can obtain the first functional relation in  $\Omega_2$  as

$$\nu_2(x) = \tau'(x) - 2\psi'(x) \quad (2.2.13)$$

We get the following result after substituting (2.2.9) and (2.2.10) into (2.2.4) and doing some evaluations:

$$\psi(x) = d_1(x) - d_2(x)\varphi(x), \quad 0 \leq x \leq 1, \quad (2.2.14)$$

where

$$d_1(z) = \frac{a_2(z) a_3(z) c_1(z) - a_1(z) a_2(z) c_3(z) + a_1(z) b_3(z) c_2(z)}{a_2(z) a_3(z) b_1(z)}$$

and

$$d_2(z) = \frac{a_1(z) b_2(z) b_3(z)}{a_2(z) a_3(z) b_1(z)}.$$

Once again substituting the solution (2.2.8) into (2.2.11) and differentiating yield that

$$\nu_2(x) = -\tau'(x) + 2\varphi(x), \quad 0 < x < 1. \quad (2.2.15)$$

From (2.2.13) and (2.2.15) one can find

$$\tau'(x) = \varphi'(x) + \psi'(x), \quad 0 < x < 1$$

We get the following result after integrating the last equality from 0 to  $x$ :

$$\tau(x) - \tau(0) = \varphi(x) - \varphi(0) + \psi(x) - \psi(0).$$

Assuming that  $a_1(0) + b_1(0) \neq 0$  and considering (2.2.9), (2.2.10) and (2.2.14) yield that

$$\begin{aligned} A_1 = \tau(0) &= \frac{c_1(0)}{a_1(0) + b_1(0)}, \\ A_2 = \tau(1) &= \frac{c_2(0)}{a_2(0)} - \frac{b_2(0)}{a_2(0)} \frac{d_1(0) - A_1}{d_2(0)}. \end{aligned} \quad (2.2.16)$$

Consequently,

$$\tau(x) = \varphi(x) + \psi(x) - \frac{d_1(0) - A_1}{d_2(0)}. \quad (2.2.17)$$

Let us assume that  $a_1(z)b_2(z)c_3(z) \neq a_2(z)a_3(z)b_1(z)$  or  $d_2(z) \neq 1$ , from the last equality we obtain

$$\varphi(z) = \frac{\tau(z) - d_1(z) - \frac{A_1 - d_1(0)}{d_2(0)}}{1 - d_2(z)}, \quad 0 \leq z \leq 1. \quad (2.2.18)$$

In short, by using (2.2.14) and (2.2.18) we can write main functional relation in  $\Omega_2$  by doing some evaluations

$$\nu_2(z) = \sigma(z)\tau'(z) + \sigma'(z)\tau(z) + e(z), \quad 0 < z < 1, \quad (2.2.19)$$

where

$$e(z) = \frac{d_2'(z)}{1 - d_2(z)} \left[ \frac{d_1(0) - A_1}{d_2(0)} - d_1(z) \right] - \frac{2d_1'(z)}{1 - d_2(z)} +$$

$$+ \frac{d_2'(z)(1+d_2(z))}{[1-d_2(z)]^2} \left[ d_1(z) - \frac{d_1(0) - A_1}{d_2(0)} \right].$$

For obtaining main functional relation in  $\Omega_1$ , we need to write the solution of (2.2.1) satisfied (2.2.6), (2.2.9), (2.2.10) conditions. In this case, we considered  $u(x, t) = \omega(x, t) + v(x, t)$ , in other words  $\omega(x, t)$  is a auxiliary function defined by

$$\omega(x, t) = g_1(t) + x[g_2(t) - g_1(t)],$$

satisfies the boundary conditions (2.2.9) and (2.2.10) and  $v(x, t)$  is the solution of the following problem

$$(\mathbf{B}) \begin{cases} v_{xx}(x, t) - {}^C(t^\theta \frac{\partial}{\partial t})^\alpha v(x, t) = q(x, t), & (x, t) \in \Omega_1, \\ v(x, 0) = \tau_0(x), & x \in [0, 1], \\ v(0, t) = v(1, t) = 0, & t \in [0, 1], \end{cases}$$

respectively, where  $\tau_0(x) = \tau(x) - \omega(x, 0)$  and  $q(x, t) = {}^C(t^\theta \frac{\partial}{\partial t})^\alpha \omega(x, t)$ .

It is obvious that from (2.2.16) and considering the boundary conditions (2.2.9) and (2.2.10) we have  $\omega(x, 0) = A_1 + x(A_2 - A_1)$ .

We note that the problem  $(\mathbf{B})$  was investigated in [5] and the existence of the solution was proved.

Considering above problems we can write the solution of the problem (2.2.1), (2.2.6), (2.2.9), (2.2.10) as follows:

$$u(x, t) = \sum_{k=1}^{\infty} \left[ \tau_{0k} E_\alpha \left( -\frac{(k\pi)^2}{p^\alpha} t^{p\alpha} \right) - \tilde{G}_k(t) \right] \sin(k\pi x) + \omega(x, t) \quad (2.2.20)$$

where  $p = 1 - \theta$  and

$$\tilde{G}_k(t) = \frac{1}{p^\alpha} \int_0^t (t^p - s^p)^{\alpha-1} E_{\alpha, \alpha} [\lambda^* (t^p - s^p)^\alpha] q_k(s) d(s^p),$$

$$q_k(t) = \frac{1}{k\pi} g_1(t) + (-1)^k \frac{1}{k\pi} g_2(t),$$

$$\tau_{0k} = 2 \int_0^1 \tau_0(x) \sin(k\pi x) dx.$$

We introduce another notation:

$$\nu_1(x) = \lim_{t \rightarrow +0} t^{1-p\alpha} u_t(x, t). \quad (2.2.21)$$

Using representation (2.2.20), we evaluate  $t^{1-p\alpha} u_t(x, t)$ :

$$\begin{aligned} t^{1-p\alpha} u_t(x, t) = \sum_{k=1}^{\infty} \left[ \tau_{0k} \left( -\frac{(k\pi)^2}{p^{\alpha-1}} \right) E_{\alpha, \alpha} \left( -\frac{(k\pi)^2}{p^{\alpha}} t^{\alpha p} \right) - \right. \\ \left. - t^{1-p\alpha} \tilde{G}'_k(t) \right] \sin(k\pi x) + t^{1-p\alpha} \omega'_t(x, t). \end{aligned}$$

Considering above evaluation, from  $1 - p\alpha > 0$  we obtain the following functional relation on  $AB$  deduced from  $\Omega_1$  as  $t \rightarrow +0$ :

$$\nu_1(x) = \frac{1}{p^{\alpha} \Gamma(\alpha)} \tau''(x). \quad (2.2.22)$$

Here, we have used  $2 \int_0^1 \tau_0(x) \sin(k\pi x) dx = -\frac{2}{(k\pi)^2} \int_0^1 \tau''(x) \sin(k\pi x) dx$ , which is true due to  $\tau_0(0) = \tau_0(1) = 0$ .

Clearly, for showing the uniqueness of the solution of the considered problem, it is enough to prove that homogeneous problem has only trivial solution.

In this case, we will have homogeneous problem at

$$c_1(z) = c_2(z) = c_3(z) = 0.$$

Then, from (2.2.16) it follows that

$$\tau(0) = \tau(1) = 0. \quad (2.2.23)$$

Also, it is comprehensible that  $e(x) = 0$  when we consider homogeneous problem.

We multiply equation (2.2.22) to the function  $\tau(x)$  and integrate along  $AB$ . Then, using integration by parts and equality (2.2.23), we get

$$\int_0^1 [\tau'(x)]^2 dx + p^\alpha \Gamma(\alpha) \int_0^1 \tau(x) \nu_1(x) dx = 0. \quad (2.2.24)$$

Let us evaluate the sign of the integral below

$$\tilde{I} = \int_0^1 \tau(x) \nu_1(x) dx. \quad (2.2.25)$$

If  $c_j(z) \equiv 0$ , ( $j = \overline{1, 3}$ ) and considering  $\nu_1(x) = \nu_2(x)$ , (2.2.25) can be written as follows:

$$\begin{aligned} \tilde{I} &= \int_0^1 \tau(x) \nu_2(x) dx = \int_0^1 \tau(x) [\sigma(x)\tau'(x) + \sigma'(x)\tau(x)] dx = \\ &= -\frac{1}{2} \int_0^1 \sigma(x) \frac{d}{dx} \tau^2(x) dx = \frac{1}{2} \int_0^1 [\tau(x)]^2 \sigma'(x) dx. \end{aligned}$$

According to Theorem 2.2.1,  $\sigma'(x) > 0$  hence,  $\tilde{I} \geq 0$ . Then, by means of (2.2.24) we find  $\tau'(x) = 0$  or  $\tau(x) = \text{const}$ .

Taking the homogeneous boundary conditions (2.2.23) into account, we determine that  $\tau(x) \equiv 0$  at  $0 \leq x \leq 1$ . From (2.2.19) and (2.2.22) we get  $\nu_1(x) = \nu_2(x) \equiv 0$ . Then according to (2.2.8) one can see that  $u(x, t) \equiv 0$  in  $\Omega_2$ . Moreover, while considering homogeneous problem, it is straightforward to find  $u(0, z) = u(1, z) = 0$ , or  $g_1(z) = g_2(z) = 0$ ,  $0 \leq z \leq 1$ . Then, from (2.2.20) it is inferred that  $u(x, t) \equiv 0$  in  $\Omega_2$ . Consequently,  $u(x, t) \equiv 0$  in  $\Omega$ , which completes the proof of uniqueness of the solution of the main problem.

### The proof of existence of the solution.

Consider the problem  $\{(2.2.22), (2.2.16)\}$ , i.e.

$$\begin{cases} \tau''(x) = p^\alpha \Gamma(\alpha) \nu_1(x), \\ \tau(0) = A_1, \quad \tau(1) = A_2. \end{cases}$$

As a result of introducing and applying a notation

$$\bar{\tau}(x) = \tau(x) - A_1 - (A_2 - A_1)x, \quad (2.2.26)$$

we get the following problem with homogeneous conditions

$$\begin{cases} \bar{\tau}''(x) = p^\alpha \Gamma(\alpha) \nu_1(x), \\ \bar{\tau}(0) = 0, \quad \bar{\tau}(1) = 0. \end{cases} \quad (2.2.27)$$

This problem can be solved using the method of Green's functions. We write the solution, considering some calculation process

$$\bar{\tau}(x) = p^\alpha \Gamma(\alpha) \int_0^1 \nu_1(\xi) G_0(x, \xi) d\xi, \quad (2.2.28)$$

where

$$G_0(x, \xi) = \begin{cases} (\xi - 1)x, & x < \xi, \\ (x - 1)\xi, & \xi < x \end{cases}$$

is Green's function of (2.2.27).

Considering the notation (2.2.26) and from (2.2.28), we get

$$\tau(x) = A_1 + (A_2 - A_1)x + p^\alpha \Gamma(\alpha) \int_0^1 \nu_1(\xi) G_0(x, \xi) d\xi. \quad (2.2.29)$$

Now considering (2.2.5) and (2.2.19), after doing some evaluations, from (2.2.29) we obtain the second kind Fredholm integral equation with respect to  $\tau(x)$ , which is equivalent to the formulated Problem F in terms of existing the solution

$$\tau(x) - \int_0^1 \tau(\xi) \tilde{K}(x, \xi) d\xi = \tilde{F}(x), \quad (2.2.30)$$

where

$$\tilde{K}(x, \xi) = -(1 - \theta)^\alpha \Gamma(\alpha) \sigma(\xi) \frac{\partial}{\partial \xi} G_0(x, \xi),$$

$$\begin{aligned} \tilde{F}(x) = & A_1 + (A_2 - A_1)x + A_1\sigma(0)x + A_2\sigma(1)(x - 1) + \\ & + (1 - \theta)^\alpha \Gamma(\alpha) \int_0^1 e(\xi) G_0(x, \xi) d\xi. \end{aligned}$$

If

$$a_i(z), b_i(z), c_i(z) \in C[0, 1] \cap C^1(0, 1), \quad i = 1, 2, 3, \quad (2.2.31)$$

then the kernel and the right side of the integral equation (2.2.30) will be continuous in their domain. Since equation (2.2.30) is equivalent to the Problem F, then from the uniqueness of the problem, it follows that equation (2.2.30) is uniquely solvable in the class of continuous functions.

Further, finding function  $\nu_2(x)$  by formula (2.2.19), we find a solution of the considered Problem F in the domain  $\Omega_2$  as the solution of the Cauchy problem by formula (2.2.8). Then, from (2.2.11), (2.2.12)  $\varphi(z)$ ,  $\psi(z)$  are found and this opens a way for finding unknown functions  $g_1(z)$ ,  $g_2(z)$ . After that we rewrite  $\omega(x, t)$  and  $\phi(x)$  with known functions. Finally, in the domain  $\Omega_1$ , solution can be represented by the formula (2.2.20) problem for equation (2.2.1) at  $t > 0$ . The proof of Theorem 2.2.1 is completed.  $\square$

**Example.** Now we give exact values to given functions and parameters in order to show the validation of the Theorem 2.2.1.

Let us say  $a_1(z) = e^z$ ,  $a_2(z) = 1$ ,  $a_3(z) = 1$ ,  $b_1(z) = 1$ ,  $b_2(z) = 2e^z + 1$ ,  $b_3(z) = e^{-z}$ ,  $c_1(z) = 1$ ,  $c_2(z) = 0$ ,  $c_3(z) = -1$  and  $\alpha = \frac{1}{2}$ ,  $\theta = \frac{3}{4}$ .

From above values of given functions, we have  $d_2(z) = 2e^z + 1$  and one can make sure that the conditions of Theorem 2.2.1 are fulfilled, i.e,  $\sigma(z) = -e^{-z} - 1$  or  $\frac{d}{dz}\sigma(z) = e^{-z} > 0$ .

## Conclusion of the Chapter 2

This Chapter consists of the results on unique solvability of the Tricomi and Frankl type problems for mixed equation involving the regularized Caputo-like counterpart of hyper-Bessel operator.

In the beginning of chapter 2 we have given the short introduction about the applications and importance of studying problems for mixed type equations.

In section 2.1 we have studied the Tricomi type problem for the mixed equation with the integral conjugation condition. The diffusion part involves the regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator which is considered in a rectangular domain while wave equation is given in characteristic triangle. The theorems for the uniqueness and the existence of the solution are proved. The method of energy integrals is used for proving the uniqueness of the solution and existence of the results showed by the reduction of the problem to the second kind Fredholm integral equation.

In section 2.2 the Frankl-type problem was at the center of investigation. Our main aim was to show a unique solvability of the considered problem. In [5] the solution was found for homogeneous boundary condition in  $\Omega_1$  only. Therefore, we needed to rewrite that solution for non-homogeneous boundary conditions. First, we use them formally, then we will find them lately. The considered problem is equivalently reduced to the second kind Fredholm integral equations. Therefore, a uniqueness of the problem we had to prove additionally. For this aim we use the method of energy integrals. All in all, sufficient conditions for given functions are presented in terms of uniqueness and existence of the solution of the Frankl type problem in the domain  $\Omega$ .

Practical value of these results can be estimated by possible application in mathematical modeling of gas movement in a channel surrounded by porous medium (see [55]).



## Chapter 3

# Non-local boundary-value problems for the mixed PDEs with the time fractional differential operators

In this chapter we deal with studying the boundary value problems for mixed differential equation consisted of sub-diffusion and fractional wave equation. While the sub-diffusion equation is generated by time fractional regularized Caputo-like counterpart of hyper-Bessel differential operator, the fractional wave equation involves the bi-ordinal Hilfer fractional derivative in time. We have to note that both fractional differential operators are generalization of other popular operators such that Caputo, Riemann-Liouville and Hadamard and hyper-Bessel operators. Sometimes they meet at some points of parameters and orders of derivative. This also impact of improving the rate of importance of these investigations.

The study of fractional order differential equations has been attracting many scientists because of its adequate and interesting applications in modeling of real-life problems related to several fields of science [36], [54]. Initial-value problems (IVPs) and boundary-value problems (BVPs) for mixed equations involving the Riemann-Liouville and Caputo derivatives attract most interest

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(see, for instance, [58], [9], [81]). Especially, studying IVPs and BVPs for the sub-diffusion, fractional wave equations are well-studied (see also [67], [96], [57], [115], [41], [56]).

We also note that in 1968, M. M. Dzhrbashyan and A. B. Nersesyan introduced the following integral-differential operator [29]

$$D_{0x}^{\sigma_n} g(x) = I_{0x}^{1-\gamma_n} D_{0x}^{\gamma_{n-1}} \dots D_{0x}^{\gamma_1} D_{0x}^{\gamma_0} g(x), \quad n \in \mathbb{N}, \quad x > 0, \quad (3.0.1)$$

which is more general than Hilfer's operator. Here  $I_{0x}^\alpha$  and  $D_{0x}^\alpha$  are the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative of order  $\alpha$  respectively (see Definition 1.3.1 and Definition 1.3.3), and  $\sigma_n \in (0, n]$  is defined by

$$\sigma_n = \sum_{j=0}^n \gamma_j - 1 > 0, \quad \gamma_j \in (0, 1].$$

There are some works [16], [3], related with this operator. New wave of investigations involving this operator might appear due to the translation of original work [29] in FCAA [30].

Considering the Remark 1.3.12, it is possible to show that Dzhrbashyan-Nersesyan fractional differential operator (3.0.1) can be reduced up to the bi-ordinal Hilfer's fractional differential operator for  $n = 1$ , i.e.

$$D_{0+}^{\sigma_1} g(t) = I_{0+}^{1-\gamma_1} D_{0+}^{\gamma_0} g(t).$$

The content of this chapter is based on the articles [122],[121] and [123], which have been already published in the journals: *Mathematical Methods in the Applied Sciences*, *Uzbek Mathematical Journal* and *Fractional Differential Calculus*.

### 3.1 Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative

In this work, we investigate a boundary value problem for a mixed equation involving the sub-diffusion equation with Caputo-like counterpart of a hyper-Bessel fractional differential operator and the fractional wave equation with Hilfer's bi-ordinal derivative in a rectangular domain. The theorem on the uniqueness and the existence of the solution has been proved. The main method is based on separation variables which is applicable and convenient to write solution explicitly rather than other methods.

In this section we present the new definitions and analogy results related to fractional hyper-Bessel differential operator and the generalized Hilfer derivative which will be used in the sequel.

#### 3.1.1 Caputo-like counterpart of the hyper-Bessel FDO with arbitrary starting point

**Definition 3.1.1.** *Regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator for  $\theta < 1$ ,  $0 < \alpha \leq 1$  and  $t > a \geq 0$  is defined in terms of the E-K fractional order operator*

$${}^C \left( (t^\theta - a^\theta) \frac{d}{dt} \right)^\alpha f(t) = (1 - \theta)^\alpha t^{-\alpha(1-\theta)} D_{1-\theta, a+}^{-\alpha, \alpha} (f(t) - f(a)) \quad (3.1.1)$$

*or in terms of the hyper-Bessel differential (R-L type) operator*

$${}^C \left( (t^\theta - a^\theta) \frac{d}{dt} \right)^\alpha f(t) = \left( (t^\theta - a^\theta) \frac{d}{dt} \right)^\alpha f(t) - \frac{f(a) \left( t^{(1-\theta)} - a^{(1-\theta)} \right)^{-\alpha}}{(1 - \theta)^{-\alpha} \Gamma(1 - \alpha)}, \quad (3.1.2)$$

where

$$\left( (t^\theta - a^\theta) \frac{d}{dt} \right)^\alpha f(t) = \begin{cases} (1 - \theta)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta, a+}^{0, -\alpha} f(t) & \text{if } \theta < 1, \\ (\theta - 1)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta, a+}^{-1, -\alpha} f(t) & \text{if } \theta > 1, \end{cases}$$

is a hyper-Bessel fractional differential operator.

From (2.1.3) for  $a = 0$  we obtain the Definition 1.3.19 presented in Section 1.3 and also Caputo FDO is the particular case of Caputo-like counterpart hyper-Bessel operator at  $\theta = 0$ .

Considering the Definition 3.1.1 we present an analogy of theorem proved in [5].

**Theorem 3.1.2.** *Assume that the following conditions hold:*

- $\tau \in C[0, 1]$  such that  $\tau(0) = \tau(1) = 0$  and  $\tau' \in L^2(0, 1)$ ,
- $f(\cdot, t) \in C^3[0, 1]$  and  $f(x, \cdot) \in C_\mu[a, T]$  such that  $f(0, t) = f(1, t) = f_{xx}(0, t) = f_{xx}(1, t) = 0$ , and  $\frac{\partial^4}{\partial x^4} f(\cdot, t) \in L^1(0, 1)$ .

Then, in  $\Omega = \{0 < x < 1, a < t < T\}$ , the problem of finding the solution of the equation

$$^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t) = f(x, t),$$

satisfying the conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad a \leq t \leq T,$$

$$u(x, a+) = \tau(x), \quad 0 \leq x \leq 1,$$

has a unique solution given by

$$u(x, t) = \sum_{k=1}^{\infty} \left[ \tau_k E_{\alpha, 1} \left( -\frac{(k\pi)^2}{p^\alpha} (t^p - a^p)^\alpha \right) + G_k(t) \right] \sin(k\pi x), \quad (3.1.3)$$

where  $p = 1 - \theta$  and

$$G_k(t) = \frac{1}{p^\alpha \Gamma(\alpha)} \int_a^t (t^p - \tau^p)^{\alpha-1} f_k(\tau) d(\tau^p) \\ - \frac{(k\pi)^2}{p^{2\alpha}} \int_a^t (t^p - \tau^p)^{2\alpha-1} E_{\alpha, 2\alpha} \left[ -\frac{(k\pi)^2}{p^\alpha} (t^p - \tau^p)^\alpha \right] f_k(\tau) d(\tau^p), \\ \tau_k = 2 \int_0^1 \tau(x) \sin(k\pi x) dx, \quad f_k(t) = 2 \int_0^1 f(x, t) \sin(k\pi x) dx, \quad k = 1, 2, 3, \dots$$

In fact, for  $a = 0$ , Theorem 3.1.2 implies the result of [5] (see Theorem 3.1).

*Proof.* It is obvious that the solution (3.1.3) satisfies the conditions. Below, we explain the derivation of the series  ${}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t)$  in (3.1.3). By using relation (3.1.2) we get:

$${}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) = \sum_{k=0}^{\infty} \left[ \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha \left( \tau_k E_{\alpha, 1} \left[ -\frac{(k\pi)^2}{p^\alpha} (t^p - a^p)^\alpha \right] + G_k(t) \right) - \frac{\tau_k (t^p - a^p)^\alpha}{p^{-\alpha} \Gamma(1 - \alpha)} \right] \sin(k\pi x).$$

The hyper-Bessel derivative of the Mittag-Leffler function is

$$\left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha \tau_k E_{\alpha, 1} \left( -\frac{(k\pi)^2}{p^\alpha} (t^p - a^p)^\alpha \right) = \tau_k p^\alpha (t^p - a^p)^{-\alpha} E_{\alpha, 1-\alpha} \left[ -(k\pi)^2 (t^p - a^p)^\alpha \right].$$

With the help of Lemma 1.4.4, we can write the last expression as

$$\left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha \tau_k E_{\alpha, 1} \left[ -\frac{(k\pi)^2}{p^\alpha} (t^p - a^p)^\alpha \right] = \frac{\tau_k p^\alpha (t^p - a^p)^{-\alpha}}{\Gamma(1 - \alpha)} + \tau_k (k\pi)^2 E_{\alpha, 1} \left[ -\frac{(k\pi)^2}{p^\alpha} (t^p - a^p)^\alpha \right].$$

Then evaluating  $\left((t^\theta - a^\theta) \frac{\partial}{\partial t}\right)^\alpha G_k(t)$  gives that

$$\begin{aligned} \left((t^\theta - a^\theta) \frac{\partial}{\partial t}\right)^\alpha G_k(t) &= \left((t^\theta - a^\theta) \frac{\partial}{\partial t}\right)^\alpha \left(f_k^*(t) + \lambda^* \int_a^t (t^p - a^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(t^p - a^p)] f_k^*(\tau) d(\tau^p)\right) \\ &= p^\alpha t^{-p\alpha} D_{p,a+}^{-\alpha,\alpha} \left(\frac{1}{p^\alpha} I_{p,a+}^{-\alpha,\alpha} t^p \alpha f_k(t) + \lambda^* \int_a^t (t^p - a^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(t^p - a^p)] f_k^*(\tau) d(\tau^p)\right) = \\ &= f_k(t) + p^\alpha t^{-p\alpha} D_{p,a+}^{-\alpha,\alpha} \left(\lambda^* \int_a^t (t^p - a^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(t^p - a^p)] f_k^*(\tau) d(\tau^p)\right), \end{aligned}$$

where  $\lambda^* = -\frac{\lambda_k}{p^\alpha}$  and  $f_k^*(t) = \frac{1}{p^\alpha \Gamma(\alpha)} \int_a^t (t^p - \tau^p)^{\alpha-1} f_k(\tau) d(\tau^p)$ .

The second term in the last expression can be simplified using the Erd'elyi-Kober fractional derivative for  $n = 1$ ,

$$\begin{aligned} & -\lambda_k t^{-p\alpha} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) \frac{t^{-p(1-\alpha)}}{\Gamma(1-\alpha)} \int_a^t (t^p - \tau^p)^{-\alpha} d(\tau^p) \times \\ & \quad \int_a^\tau (\tau^p - s^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(\tau^p - s^p)^\alpha] f_k^*(s) d(s^p) = \\ & -\lambda_k t^{-p\alpha} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) \frac{t^{-p(1-\alpha)}}{\Gamma(1-\alpha)} \int_a^t f_k^*(s) d(s^p) \times \\ & \quad \int_s^t (t^p - \tau^p)^{-\alpha} (\tau^p - s^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(\tau^p - s^p)^\alpha] d(\tau^p) = \\ & -\lambda_k t^{-p\alpha} \left(1 - \alpha + \frac{t}{p} \frac{d}{dt}\right) t^{-p(1-\alpha)} \int_a^t E_{\alpha,1}[\lambda^*(t^p - s^p)^\alpha] f_k^*(s) d(s^p) = \\ & \quad -\lambda_k (1 - \alpha) t^{-p} \int_a^t E_{\alpha,1}[\lambda^*(t^p - s^p)^\alpha] f_k^*(s) d(s^p) - \\ & \quad -\frac{\lambda_k t^{-p\alpha+1}}{p} \frac{d}{dt} \left(t^{-p(1-\alpha)} \int_a^t E_{\alpha,1}[\lambda^*(t^p - s^p)^\alpha] f_k^*(s) d(s^p)\right) = \\ & \quad = -\lambda_k (1 - \alpha) t^{-p} \int_a^t E_{\alpha,1}[\lambda^*(t^p - s^p)^\alpha] f_k^*(s) d(s^p) + \\ & \quad + \lambda_k (1 - \alpha) t^{-p} \int_a^t E_{\alpha,1}[\lambda^*(t^p - \tau^p)^\alpha] f_k^*(\tau) d(\tau^p) - \lambda_k f_k^*(t) - \\ & \quad -\lambda_k t^{1-p} \int_a^t \lambda^*(t^p - \tau^p)^{\alpha-1} E_{\alpha,\alpha}[\lambda^*(t^p - \tau^p)^\alpha] f_k^*(\tau) d(\tau^p) = \end{aligned}$$

$$= -\lambda_k \left( f_k^*(t) + \lambda^* \int_a^t (t^p - a^p)^{\alpha-1} E_{\alpha,\alpha} [\lambda^* (t^p - a^p)] f_k^*(\tau) d(\tau^p) \right) = -\lambda_k G_k(t).$$

Hence, we get

$$\begin{aligned} {}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) = \\ - \sum_{k=0}^{\infty} (k\pi)^2 \left[ \tau_k E_{\alpha,1} \left( \frac{(k\pi)^2}{p^\alpha} (t^p - a^p) \right) + G_k(t) \right] \sin(k\pi x) + f(x, t). \end{aligned}$$

This proves that the solution (3.1.3) satisfies the equation

$${}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t) = f(x, t).$$

□

### 3.1.2 Differential equation involving bi-ordinal Hilfer derivative

As presented in Definition 1.3.11, the bi-ordinal Hilfer's fractional derivative of orders  $\gamma \in (1, 2]$ , and  $\beta \in (1, 2]$  and type  $\mu \in [0, 1]$  can be written as a special case of (1.3.4) for  $n = 2$ :

$$D_t^{(\gamma,\beta)\mu} f(t) = I_{0+}^{\mu(2-\gamma)} \left( \frac{d}{dt} \right)^2 I_{0+}^{(1-\mu)(2-\beta)} f(t). \quad (3.1.4)$$

Here we present the formula for the Laplace transform of (3.1.4) for the convenience of the reader which will be used later:

$$\begin{aligned} \mathcal{L}\{D_t^{(\alpha,\beta)\mu} f(t)\} &= s^{\beta+\mu(\alpha-\beta)} \mathcal{L}\{f(t)\} - s^{1-\mu(2-\alpha)} \times \\ &\times \left[ I_{0+}^{(1-\mu)(2-\beta)} f(t)|_{t \rightarrow 0+} \right] - s^{-\mu(2-\alpha)} \left[ \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} f(t)|_{t \rightarrow 0+} \right]. \end{aligned} \quad (3.1.5)$$

Let us consider the following problem:

*Find a solution of the equation*

$$D_t^{(\gamma,\beta)\mu} y(t) + \lambda y(t) = f(t), \quad (1 < \gamma, \beta \leq 2, \quad 0 \leq \mu \leq 1), \quad (3.1.6)$$

*satisfying the initial conditions*

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(2-\beta)} y(t) = \xi_0, \quad (3.1.7)$$

$$\lim_{t \rightarrow 0+} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} y(t) = \xi_1, \quad (3.1.8)$$

where  $f(t)$  is a given function and  $\lambda, \xi_0, \xi_1 = \text{const}$ .

We note that in [49] R. Hilfer et al. considered the Cauchy problem with fractional differential equation involving the generalized Riemann-Liouville fractional derivative of order  $n-1 < \alpha < n, n \in \mathbb{N}$  and using operational method they presented explicit solution.

**Lemma 3.1.3.** *If  $f \in C^1_{-1}(0, +\infty)$ , then the problem (3.1.6)-(3.1.8) has an unique solution represented by*

$$\begin{aligned} y(t) = & \xi_0 t^{(\beta-2)(1-\mu)} E_{\delta, \delta+\mu(2-\gamma)-1}(-\lambda t^\delta) + \xi_1 t^{\mu+(\beta-1)(1-\mu)} E_{\delta, \delta+\mu(2-\gamma)}(-\lambda t^\delta) + \\ & + \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta}(-\lambda(t-\tau)^\delta) f(\tau) d\tau, \end{aligned} \quad (3.1.9)$$

where  $\delta = \beta + \mu(\gamma - \beta)$ .

*Proof.* In fact, applying the Laplace transform to (3.1.6) by means of (3.1.5) and considering initial conditions (3.1.7), (3.1.8) yield

$$\mathcal{L}\{u\} = \frac{\xi_0 s^{1-\mu(2-\gamma)} + \xi_1 s^{-\mu(2-\gamma)} + \mathcal{L}\{f\}}{s^{\beta+\mu(\gamma-\beta)} + \lambda}, \quad (3.1.10)$$

where  $\mathcal{L}\{u\}$  and  $\mathcal{L}\{f\}$  are the Laplace transform of functions  $u$  and  $f$ , respectively.

According to Lemma 1.4.3, the Laplace transform of the Mittag-Leffler function can be written as follows

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^{1-\mu(2-\gamma)}}{s^{\beta+\mu(\gamma-\beta)} + \lambda} \right\} &= t^{\beta-2+\mu(2-\beta)} E_{\beta+\mu(\gamma-\beta), \beta-1+\mu(2-\beta)}(-\lambda t^{\beta+\mu(\gamma-\beta)}), \\ \mathcal{L}^{-1} \left\{ \frac{s^{-\mu(2-\gamma)}}{s^{\beta+\mu(\gamma-\beta)} + \lambda} \right\} &= t^{\beta-1+\mu(2-\beta)} E_{\beta+\mu(\gamma-\beta), \beta+\mu(2-\beta)}(-\lambda t^{\beta+\mu(\gamma-\beta)}), \\ \mathcal{L}^{-1} \left\{ \frac{\mathcal{L}\{f\}}{s^{\beta+\mu(\gamma-\beta)} + \lambda} \right\} &= \\ \int_0^t (t-\tau)^{\beta-1+\mu(\gamma-\beta)} E_{\beta+\mu(\gamma-\beta), \beta+\mu(\gamma-\beta)}(-\lambda(t-\tau)^{\beta+\mu(\gamma-\beta)}) f(\tau) d\tau, \end{aligned}$$



where  $\mathcal{L}^{-1}$  is an inverse Laplace transform operator.

Considering above evaluations and after applying the inverse Laplace transform to (3.1.10), we can write the solution of (3.1.6)- (3.1.8) in the form (3.1.9).  $\square$

### 3.1.3 The statement of the main problem and its investigation

Let us consider the following equation

$$f(x, t) = \begin{cases} {}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_1, \\ D_t^{(\gamma, \beta)\mu} u(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_2, \end{cases} \quad (3.1.11)$$

in a domain  $\Omega = \Omega_1 \cup \Omega_2 \cup Q$ . Here  $\Omega_1 = \{(x, t) : 0 < x < 1, a < t < b\}$ ,  $\Omega_2 = \{(x, t) : 0 < x < 1, 0 < t < a\}$ ,  $Q = \{(x, t) : 0 < x < 1, t = a\}$ ,  $a, b \in \mathbb{R}^+$  such that  $a < b$ ,  $0 < \alpha \leq 1$ ,  $\theta < 1$ ,  $1 < \gamma, \beta < 2$ ,  $0 \leq \mu \leq 1$ ,  $f(x, t)$  is a given function,  ${}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha$  is the regularized Caputo-like counterpart of the hyper-Bessel operator defined as in (3.1.1), (3.1.2)  $D_t^{(\gamma, \beta)\mu}$  is the bi-ordinal Hilfer's derivative defined as in (3.1.4).

**Problem.** Find a solution of (3.1.11) in  $\Omega$ , satisfying regularity conditions

$$\begin{aligned} u(\cdot, t) &\in C[0, 1] \cap C^1(0, 1), \quad {}^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, \cdot) \in C[a, b], \\ t^{2-q}u, \quad t^{2-q}u_x &\in C(\overline{\Omega}_2), \quad t^{2-q}D_t^{(\gamma, \beta)\mu}u(x, t) \in C(\overline{\Omega}_2), \quad u_{xx} \in C(\Omega) \end{aligned}$$

and the boundary-initial conditions

$$u(0, t) = 0, \quad 0 \leq t \leq b, \quad (3.1.12)$$

$$u(1, t) = 0, \quad 0 \leq t \leq b, \quad (3.1.13)$$

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(2-\beta)} u(x, t) = \varphi(x), \quad 0 \leq x \leq 1, \quad (3.1.14)$$

as well as the gluing conditions

$$\lim_{t \rightarrow a-} I_{0+}^{(1-\mu)(2-\beta)} u(x, t) = \lim_{t \rightarrow a+} u(x, t), \quad 0 \leq x \leq 1, \quad (3.1.15)$$

$$\lim_{t \rightarrow a-} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} u(x, t) = \lim_{t \rightarrow a+} (t - a)^{1-(1-\theta)\alpha} u_t(x, t), \quad 0 < x < 1, \quad (3.1.16)$$

where  $\varphi(x)$  is a given function,  $q = \beta + \mu(2 - \beta)$ .

The key motivation to formulate this problem is a possible application in diffusion-wave processes, which will be described by the mixed equation as Eq.(3.1.11) [55]. For example, a gas movement in a channel surrounded by porous medium will be governed by the mixed parabolic-hyperbolic type equation, because inside of the channel movement will be described by the wave equation, in porous media by diffusion equation [42]. Practical importance of the diffusion part of the considered mixed equation can be seen in [39]. Regarding the remained part we note that use bi-ordinal Hilfer derivative generalizes classical wave equation and both two fractional generalizations: the Riemann-Liouville and the Caputo cases.

Our choice of the method of separation of variables is motivated by the considered domain which allows us to use this powerful method. We, as well, note that there is a method of Green's function, which is successfully applied for fractional diffusion-wave equations by A.Pskhu [86], [88]. However, in our case, there are certain difficulties linked to the unknown properties of the hyper-Bessel operator which did not allow us to use this tool.

First we introduce the following new notations:

$$\lim_{t \rightarrow 0+} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} u(x, t) = \psi(x), \quad 0 < x < 1, \quad (3.1.17)$$

$$\lim_{t \rightarrow a+} u(x, t) = \tau(x), \quad 0 \leq x \leq 1, \quad (3.1.18)$$

here  $\tau(x)$  and  $\psi(x)$  are unknown functions to be found later.

Using the method of separation of variables for solving the homogeneous equation corresponding to (3.1.11), i.e. searching for a solution as  $u(x, t) = T(t)X(x)$  and considering (3.1.12) and (3.1.13) in homogeneous case, yield the

following problem:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(1) = 0. \quad (3.1.19)$$

It is obvious that (3.1.19) is a Sturm-Liouville problem on finding eigenvalues and eigenfunctions and it has the following solution:

$$\lambda_k = (k\pi)^2, \quad X_k(x) = \sin(k\pi x), \quad k = 1, 2, 3, \dots \quad (3.1.20)$$

Considering the fact that the system of eigenfunctions  $\{X_k(x)\}$  in (3.1.20) forms the orthogonal basis in  $L^2(0, 1)$  [79], we look for the solution  $u(x, t)$  and given function  $f(x, t)$  in the form of series expansions as follows:

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x), \quad (3.1.21)$$

$$f(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x), \quad (3.1.22)$$

where  $u_k(t)$  is unknown function to be found,  $f_k(t)$  are known and given by

$$f_k(t) = 2 \int_0^1 f(x, t) \sin(k\pi x) dx.$$

Substituting (3.1.21) and (3.1.22) into equation (3.1.11) in  $\Omega_1$  and considering initial condition (3.1.18) gives the following fractional differential equation

$${}^C \left( (t^\theta - a^\theta) \frac{d}{dt} \right)^\alpha u_k(t) + (k\pi)^2 u_k(t) = f_k(t)$$

with initial condition

$$u_k(a+) = \tau_k,$$

where  $\tau_k$  is the coefficient of series expansion of  $\tau(x)$  in terms of orthogonal basis (3.1.20), i.e.,

$$\tau_k = 2 \int_0^1 \tau(x) \sin(k\pi x) dx.$$

After finding the solution of this problem, then considering (3.1.21) we can write the solution of (3.1.11) in  $\Omega_1$  satisfying the conditions (3.1.12), (3.1.13) and (3.1.18) stated in (3.1.3).

Now by using the solution (3.1.3), we evaluate  $(t - a)^{1-(1-\theta)\alpha}u_t(x, t)$ :

$$(t - a)^{1-(1-\theta)\alpha}u_t(x, t) = \sum_{k=1}^{\infty} \left[ -\frac{(k\pi)^2}{p^{\alpha-1}} \tau_k E_{\alpha, \alpha} \left( -\frac{(k\pi)^2}{p^{\alpha}} (t^p - a^p)^{\alpha} \right) + (t - a)^{1-p\alpha} G_k(t) \right] \sin(k\pi x),$$

where  $p = 1 - \theta$ .

Considering above given evaluations we obtain the following relation on  $Q$  deduced from  $\Omega_1$  as  $t \rightarrow a+$ :

$$\lim_{t \rightarrow a+} (t - a)^{1-(1-\theta)\alpha}u_t(x, t) = \sum_{k=1}^{\infty} \left[ -\frac{(k\pi)^2}{\Gamma(\alpha)p^{\alpha-1}} \tau_k \right] \sin(k\pi x). \quad (3.1.23)$$

Now we establish another relation on  $Q$  which will be reduced from  $\Omega_2$ .

According to the method of separation of variables, considering (3.1.21), (3.1.22) and initial conditions (3.1.14), (3.1.17), we obtain the following problem finding a solution of equation

$$D_t^{(\gamma, \beta)\mu} u_k(t) + \lambda_k u_k(t) = f_k(t),$$

satisfying the initial conditions

$$\begin{aligned} \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(2-\beta)} u_k(t) &= \varphi_k, \\ \lim_{t \rightarrow 0+} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} u_k(t) &= \psi_k. \end{aligned}$$

According to Lemma 3.1.3 it is obvious that (3.1.9) is the solution for above given problem. Hence, using the solution (3.1.9) and taking (3.1.21) into account we write the solution of (3.1.11) in  $\Omega_2$  satisfying (3.1.12), (3.1.13) and (3.1.14), (3.1.17) as

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x), \quad (3.1.24)$$

where

$$u_k(t) = \varphi_k t^{(\beta-2)(1-\mu)} E_{\delta, \delta+\mu(2-\gamma)-1}(-\lambda_k t^\delta) + \psi_k t^{\mu+(\beta-1)(1-\mu)} E_{\delta, \delta+\mu(2-\gamma)}(-\lambda_k t^\delta) + \\ + \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta}[-\lambda_k(t-\tau)^\delta] f_k(\tau) d\tau,$$

here  $\delta = \beta + \mu(\gamma - \beta)$  and

$$\varphi_k = 2 \int_0^1 \varphi(x) \sin(k\pi x) dx$$

and  $\psi_k$  are not known yet.

Now using (3.1.24) we simplify  $\lim_{t \rightarrow a-} I_{0+}^{(1-\mu)(2-\beta)} u_k(t)$  and  $\lim_{t \rightarrow a-} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} u_k(t)$  as follows

$$\lim_{t \rightarrow a-} I_{0+}^{(1-\mu)(2-\beta)} u_k(t) = \varphi_k E_{\delta, 1}(-\lambda_k a^\delta) + \psi_k a E_{\delta, 2}(-\lambda_k a^\delta) + \int_0^a (a-s)^{\delta+q-1} E_{\delta, \delta+q}[-\lambda_k(a-s)^\delta] f_k(s) ds, \quad (3.1.25)$$

$$\lim_{t \rightarrow a-} \frac{d}{dt} I_{0+}^{(1-\mu)(2-\beta)} u_k(t) = -\varphi_k \lambda_k a^{\delta-1} E_{\delta, \delta}(-\lambda_k a^\delta) + \psi_k E_{\delta, 1}(-\lambda_k a^\delta) + \int_0^a (a-s)^{\delta+q-2} E_{\delta, \delta+q-1}[-\lambda_k(a-s)^\delta] f_k(s) ds. \quad (3.1.26)$$

After substituting (3.1.25) and (3.1.18) into gluing condition (3.1.15) and substituting (3.1.26), (3.1.23) into the gluing condition (3.1.16), we obtain the following the system of linear algebraic equations with respect to  $\tau_k$  and  $\psi_k$ :

$$\left\{ \begin{array}{l} \varphi_k E_{\delta, 1}(-\lambda_k a^\delta) + \psi_k a E_{\delta, 2}(-\lambda_k a^\delta) + \\ + \int_0^a (a-s)^{\delta+q-1} E_{\delta, \delta+q}[-\lambda_k(a-s)^\delta] f_k(s) ds = \tau_k, \\ \varphi_k \lambda_k a^{\delta-1} E_{\delta, \delta}(-\lambda_k a^\delta) - \psi_k E_{\delta, 1}(-\lambda_k a^\delta) - \\ - \int_0^a (a-s)^{\delta+q-2} E_{\delta, \delta+q-1}[-\lambda_k(a-s)^\delta] f_k(s) ds = \frac{\lambda_k}{\Gamma(\alpha) p^{\alpha-1}} \tau_k. \end{array} \right. \quad (3.1.27)$$

From (3.1.27), we find  $\psi_k$  and  $\tau_k$ :

$$\psi_k = \frac{B_k}{\Delta_k} \varphi_k + \frac{C_k}{\Delta_k}, \quad (3.1.28)$$

$$\begin{aligned} \tau_k = & \left( E_{\delta,1}(-\lambda_k a^\delta) + \frac{B_k}{\Delta_k} E_{\delta,2}(-\lambda_k a^\delta) \right) \varphi_k + \frac{C_k}{\Delta_k} E_{\delta,2}(-\lambda_k a^\delta) + \\ & + \int_0^a (a-s)^{\delta+q-1} E_{\delta,\delta+q}[-\lambda_k(a-s)^\delta] f_k(s) ds, \end{aligned} \quad (3.1.29)$$

where

$$\begin{aligned} \Delta_k &= E_{\delta,1}(-\lambda_k a^\delta) + \frac{\lambda_k a}{\Gamma(\alpha) p^{\alpha-1}} E_{\delta,2}(-\lambda_k a^\delta), \\ B_k &= \frac{-\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} E_{\delta,1}(-\lambda_k a^\delta) + \lambda_k a^{\delta-1} E_{\delta,\delta}(-\lambda_k a^\delta), \\ C_k &= \frac{-\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \int_0^a (a-s)^{\delta+q-1} E_{\delta,\delta+q}[-\lambda_k(a-s)^\delta] f_k(s) ds - \\ & - \int_0^a (a-s)^{\delta+q-2} E_{\delta,\delta+q-1}[-\lambda_k(a-s)^\delta] f_k(s) ds, \end{aligned}$$

here  $\lambda_k = (k\pi)^2$ ,  $q = (1-\mu)(2-\beta)$ .

First of all, we will find an estimate for  $B_k$  by using Lemma 1.4.1:

$$\begin{aligned} |B_k| &\leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} |E_{\delta,1}(-\lambda_k a^\delta)| + \lambda_k a^{\delta-1} |E_{\delta,\delta}(-\lambda_k a^\delta)| \leq \\ &\leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \frac{M}{1 + \lambda_k a^\delta} + \lambda_k a^{\delta-1} \frac{M}{1 + \lambda_k a^\delta} \leq \\ &\leq \frac{\lambda_k p^{1-\alpha}}{\Gamma(\alpha)} \frac{M}{\lambda_k a^\delta} + \lambda_k a^{\delta-1} \frac{M}{\lambda_k a^\delta} = \frac{M p^{1-\alpha}}{a^\delta \Gamma(\alpha)} + \frac{M}{a} = \\ &= \frac{M}{a} \left( 1 + \frac{p^{1-\alpha}}{a^{\delta-1}} \right) = M_1 < \infty, \quad (M_1 = \text{const}). \end{aligned}$$

Now let us find the upper bound of  $C_k$  after integrating by parts the integrals in it:

$$\begin{aligned} |C_k| &\leq \frac{|-\lambda_k| p^{1-\alpha}}{\Gamma(\alpha)} \int_0^a |a-s|^{\delta+q-1} |E_{\delta,\delta+q}(-\lambda_k(a-s)^\delta)| |f_k(s)| ds + \\ &+ |f_k(0)| a^{\delta+q} |E_{\delta,\delta+q}[-\lambda_k a^\delta]| + \int_0^a |a-s|^{\delta+q-1} |E_{\delta,\delta+q}(-\lambda_k(a-s)^\delta)| |f'_k(s)| ds \leq \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^a |a-s|^{\delta+q-1} \left| \frac{M}{1+\lambda_k|a-s|^\delta} \right| |f_{2k}(s)| ds + |f_k(0)| \frac{a^{\delta+q}M}{1+\lambda_k a^\delta} + \\
 &\quad + \int_0^a |a-s|^{\delta+q-1} \left| \frac{M}{1+\lambda_k|a-s|^\delta} \right| |f'_k(s)| ds \leq \\
 &\leq \frac{p^{1-\alpha}}{\lambda_k \Gamma(\alpha)} \int_0^a |a-s|^{q-1} M |f_{2k}(s)| ds + \frac{|f_k(0)| M a^q}{\lambda_k} + \int_0^a |a-s|^{q-1} \frac{M |f'_k(s)|}{\lambda_k} ds = \\
 &= \frac{1}{\lambda_k} \left[ \frac{p^{1-\alpha} M C_1 a^q}{\Gamma(\alpha) q} + |f_k(0)| M a^q + \frac{M C_2 a^q}{q} \right] = \frac{M_2}{\lambda_k}, \quad M_2 > 0,
 \end{aligned}$$

where  $|f_{2k}(t)| \leq \overline{C}_1$ ,  $|f'_k(t)| \leq \overline{C}_2$ ,  $f_{2k}(t) = -\lambda_k f_k(t)$ ,  $f'_k(t) = 2 \int_0^1 f_t(x, t) \sin(k\pi x) dx$ .

Note that on above inequalities we imply that  $f(\cdot, t) \in C^1(0, 1)$ ,  $f_{xx}(\cdot, t) \in L^1(0, 1)$  and  $f(x, \cdot) \in C^1(0, a)$  for convergence of the last integrals.

By using above evaluations, we find the estimate for  $|\psi_k|$ :

$$\begin{aligned}
 |\psi_k| &\leq \frac{1}{|\Delta_k|} \left[ |B_k| |\varphi_k| + |C_k| \right] \leq \frac{1}{|\Delta_k|} \left[ \frac{M_1 |\varphi_{1k}|}{\sqrt{\lambda_k}} + \frac{M_2}{\lambda_k} \right] \leq \\
 &\leq \frac{1}{|\Delta_k|} \left( \frac{2M_1^2}{\lambda_k} + 2|\varphi_{1k}|^2 + \frac{M_2}{\lambda_k} \right) = \frac{M_3}{\lambda_k}
 \end{aligned} \tag{3.1.30}$$

where  $M_3 > 0$ ,  $\varphi_{1k} = -\sqrt{\lambda_k} \varphi_k$  and we assume  $\varphi' \in L^2(0, 1)$ , provided that  $\Delta_k \neq 0$  for any  $k$ .

From (3.1.29) and in the same way one can show that

$$\begin{aligned}
 |\tau_k| &\leq \left[ |E_{\delta,1}(-\lambda_k a^\delta)| + \left| \frac{B_k}{\Delta_k} |E_{\delta,2}(-\lambda_k a^\delta)| \right] |\varphi_k| + \left| \frac{C_k}{\Delta_k} |E_{\delta,2}(-\lambda_k a^\delta)| \right| + \\
 &\quad + \int_0^a |a-s|^{\delta+q-1} |E_{\delta,\delta+q}(-\lambda_k(a-s)^\delta)| |f_k(s)| ds \leq \\
 &\leq \left[ \frac{M}{1+\lambda_k a^\delta} + \left| \frac{B_k}{\Delta_k} \right| \frac{M}{1+\lambda_k a^\delta} \right] |\varphi_k| + \left| \frac{C_k}{\Delta_k} \right| \frac{M}{1+\lambda_k a^\delta} + \\
 &\quad + \int_0^a |a-s|^{\delta+q-1} \frac{M}{1+\lambda_k|a-s|^\delta} |f_k(s)| ds \leq
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{M|\varphi_k|}{\lambda_k a^\delta} \left(1 + \left|\frac{B_k}{\Delta_k}\right|\right) + \left|\frac{C_k}{\Delta_k}\right| \frac{M}{\lambda_k a^\delta} + \frac{M}{\lambda_k} \int_0^a |a-s|^{q-1} |f_k(s)| ds \leq \\
 &\leq \frac{1}{\lambda_k} \left[ \frac{M}{a^\delta} \left( |\varphi_k| + \left|\frac{B_k}{\Delta_k}\right| |\varphi_k| + \left|\frac{C_k}{\Delta_k}\right| \right) + \frac{M C_0 a^q}{q} \right] = \frac{M_4}{\lambda_k}, \tag{3.1.31}
 \end{aligned}$$

where  $M_4 > 0$ ,  $\varphi \in L^1(0, 1)$ ,  $|f_k(t)| \leq \overline{C_0}$ .

The system of linear equations (3.1.27) is equivalent to the considered problem in terms of existing the solution. For that reason, if  $\Delta_k \neq 0$  for any  $k$ , (3.1.27) has only one solution or the considered problem's solution is unique, if it exists. Therefore, we show that  $\Delta_k$  is not equal to zero for any sufficiently large  $k$ .

By using Lemma 1.4.5, the behavior of  $\Delta_k$  at  $k \rightarrow \infty$  can be written as:

$$\lim_{k \rightarrow \infty} \Delta_k = \lim_{|z| \rightarrow \infty} \left[ E_{\delta,1}(z) + \frac{p^{1-\alpha}}{\Gamma(\alpha) a^{\delta-1}} z E_{\delta,2}(z) \right] = \frac{p^{1-\alpha}}{\Gamma(\alpha) \Gamma(2-\delta) a^{\delta-1}},$$

where  $z = -\lambda_k a^\delta$ . This proves that  $\Delta_k \neq 0$  for sufficiently large  $k$ .

For proving the existence of the solution, we need to show uniform convergence of series representations of  $u(x, t)$ ,  $u_{xx}(x, t)$ ,  ${}^C(t^\theta \frac{\partial}{\partial t})^\alpha u(x, t)$  and  $D_t^{(\gamma, \beta)\mu} u(x, t)$  by using the solution (3.1.3) and (3.1.24) in  $\Omega_1$  and  $\Omega_2$  respectively.

In [5], the uniform convergence of series of  $u(x, t)$  and  $u_{xx}(x, t)$  was shown for  $t > 0$ . Similarly, for  $t > a$ , we obtain the following estimate:

$$\begin{aligned}
 |u(x, t)| \leq M \sum_{k=1}^{\infty} &\left( \frac{|\tau_k|}{p^\alpha + (k\pi)^2 |t^p - a^p|^\alpha} + \frac{1}{(k\pi)^2} \int_a^t |t^p - \tau^p|^{\alpha-1} f_{2k}(\tau) d(\tau^p) + \right. \\
 &\left. + \int_a^t \frac{|t^p - \tau^p|^{2\alpha-1}}{p^\alpha + (k\pi)^2 |t^p - a^p|^\alpha} |f_{2k}(\tau)| d(\tau^p) \right),
 \end{aligned}$$

where  $|f_{2k}(t)| \leq \overline{C_1}$ ,  $f_{2k}(t) = 2 \int_0^1 f_{xx}(x, t) \sin(k\pi x) dx$ .

Since (3.1.31) and  $\frac{\partial^2}{\partial x^2} f(\cdot, t) \in L^1(0, 1)$ , then the above series converges and hence, by the Weierstrass M-test the series of  $u(x, t)$  is uniformly convergent in  $\Omega_1$ .



The series of  $u_{xx}(x, t)$  is written in the form below

$$u_{xx}(x, t) = - \sum_{k=1}^{\infty} (k\pi)^2 \left( \tau_k E_{\alpha,1} \left[ \frac{(k\pi)^2}{p^\alpha} (t-a)^{p\alpha} \right] + G_k(t) \right) \sin(k\pi x).$$

We obtain the following estimate by considering the inequality (3.1.31):

$$|u_{xx}(x, t)| \leq M \sum_{k=1}^{\infty} \left( \frac{M_4}{p^\alpha + (k\pi)^2 |t^p - y^p|^\alpha} + \frac{1}{(k\pi)^2} \int_a^t |t^p - \tau^p|^{\alpha-1} |f_{4k}(\tau)| d(\tau^p) \right. \\ \left. + \int_a^t \frac{|t^p - \tau^p|^{2\alpha-1}}{p^\alpha + (k\pi)^2 |t^p - \tau^p|^\alpha} |f_{4k}(\tau)| d(\tau^p) \right),$$

where  $f_{4k}(t) = 2 \int_0^1 \frac{\partial^4}{\partial x^4} f(x, t) \sin(k\pi x) dx$  and  $f(0, t) = f(1, t) = f_{xx}(0, t) = f_{xx}(1, t) = 0$ .

Since  $\frac{\partial^4 f}{\partial x^4}(\cdot, t) \in L^1(0, 1)$ , one can make sure that this above series is convergent.

Thus, the series in the expression of  $u_{xx}(x, t)$  is bounded by a convergent series which is uniformly convergent according to the Weierstrass M-test. Then, the series of  $^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t)$  which can be written by

$$^C \left( (t^\theta - a^\theta) \frac{\partial}{\partial t} \right)^\alpha u(x, t) = \\ - \sum_{k=1}^{\infty} (k\pi)^2 \left( \tau_k E_{\alpha,1} \left[ - \frac{(k\pi)^2}{p^\alpha} (t-a)^{p\alpha} \right] + G_k(t) \right) \sin(k\pi x) + f(x, t),$$

has uniform convergence which can be showed in the same way to the uniform convergence of the series of  $u_{xx}(x, t)$  (see [5]).

Now we need to show that the series of  $t^{2-q}u(x, t)$  and its derivatives should converge uniformly in  $\Omega_2$  by using (3.1.24). We estimate that

$$|t^{2-q}u(x, t)| \leq \sum_{k=1}^{\infty} \left( |\varphi_k| |E_{\delta, \delta+\mu(2-\gamma)-1}(-\lambda_k t^\delta)| + |\psi_k| |t E_{\delta, \delta+\mu(2-\gamma)}(-\lambda_k t^\delta)| + \right.$$

$$+|f_k(0)||t^{\delta+2-q}|E_{\delta,\delta+1}(-\lambda_k t^\delta)|+t^{2-q}\int_0^t|t-\tau|^\delta E_{\delta,\delta+1}\left(-\lambda_k(t-\tau)^\delta\right)||f'_k(\tau)|d\tau).$$

Consider estimates of the Mittag-Leffler function (see Lemma 1.4.1)

$$|t^{2-q}u(x,t)|\leq\sum_{k=1}^{\infty}\left(\frac{|\varphi_k|M}{1+\lambda_k t^\delta}+\frac{|t||\psi_k|M}{1+\lambda_k t^\delta}+\frac{|f_k(0)||t^{\delta+2-q}|M}{1+\lambda_k t^\delta}+t^{2-q}\int_0^t|t-\tau|^\delta\frac{M}{1+\lambda_k|t-\tau|^\delta}|f'_k(\tau)|d\tau\right),$$

where  $f'_k(t)=\int_0^1 f'_t(x,t)\sin(k\pi x)dx$ ,  $f'_t(\cdot,t)\in L^1(0,1)$  and  $\varphi'(x)\in L^2(0,1)$ .

Then the series is convergent or in other words the functional series represents  $|u(x,t)|$  is bounded by the convergent series. According to the Weierstrass M-test this functional series converges uniformly in  $\Omega_2$ .

In the similar way one can show that

$$|u_{xx}(x,t)|\leq\sum_{k=1}^{\infty}(k\pi)^2\left(|\varphi_k||t^{(\beta-2)(1-\mu)}E_{\delta,\delta+\mu(2-\gamma)-1}(-\lambda_k t^\delta)|+|t^{\mu+(\beta-1)(1-\mu)}E_{\delta,\delta+\mu(2-\gamma)}(-\lambda_k t^\delta)|+|f_k(0)||t^\delta|E_{\delta,\delta+1}(-\lambda_k t^\delta)|+\int_0^t|t-\tau|^\delta E_{\delta,\delta+1}\left(-\lambda_k(t-\tau)^\delta\right)||f'_k(\tau)|d\tau\right).$$

Considering (3.1.30), and the properties of Mittag-Leffler function we write the following estimate for  $u_{xx}(x,t)$  in  $\Omega_2$

$$|u_{xx}(x,t)|\leq\sum_{k=1}^{\infty}\left[\frac{|\varphi_k|M\lambda_k t^{(\beta-2)(1-\mu)}}{1+\lambda_k t^\delta}+\frac{M_3 t^{\mu+(\beta-1)(1-\mu)}}{1+\lambda_k t^\delta}+\frac{M||f_{2k}(0)t^\delta|}{1+\lambda_k a^\delta}+\int_0^t|t-\tau|^\delta\frac{M}{1+\lambda_k(t-\tau)^\delta}|f'_{2k}(\tau)|d\tau\right],$$

where  $f'_{2k}(t)=2\int_0^1 f_{xxt}(x,t)\sin(k\pi x)dx$ , and also it implies that  $f(x,\cdot)\in C^1(0,a)$ ,  $f_{xxt}(\cdot,t)\in L^1(0,1)$  and  $\varphi'\in L^2(0,1)$ . Here we also considered that

$$\sum_{k=1}^{\infty}|\varphi_k|=\sum_{k=1}^{\infty}\frac{|\varphi_{1k}|}{k\pi}\leq\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{(k\pi)^2}+|\varphi_{1k}|^2\right)=\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}+\|\varphi'\|_{L^2(0,1)}.$$

From the last inequality, we can see that the functional series  $u_{xx}(x, t)$  is bounded by convergent series and it means that this functional series converges uniformly in  $\Omega_2$ , according to the Weierstrass M-test.

Using the equation in  $\Omega_2$ , we write  $D_t^{(\gamma, \beta)\mu} u(x, t)$  in the form

$$D_t^{(\gamma, \beta)\mu} u(x, t) = u_{xx}(x, t) + f(x, t)$$

and its uniform convergence can be shown in a similar way.

Finally, considering the Weierstrass M-test, the above arguments prove that Fourier series in (3.1.3) and (3.1.24) converge uniformly in the domains  $\Omega_1$  and  $\Omega_2$ . This is the proof that the considered problem's solution exists in  $\Omega$ .

The intention of this paper was to prove the uniqueness and existence of the solution to the problem (3.1.11)-(3.1.16), as we summarize in the following theorem.

**Theorem 3.1.4.** *If the following conditions*

- 1)  $\Delta_k \neq 0$ , for all  $k = 1, 2, 3, \dots$
- 2)  $\varphi \in C[0, 1]$  and  $\varphi' \in L^2(0, 1)$ ,
- 3)  $f(\cdot, t) \in C^3[0, 1]$  and  $f(x, \cdot) \in C^1(0, a)$ ,  $f(x, \cdot) \in C_\mu(a, b)$ , such that  $f(0, t) = f(1, t) = 0$ ,  $f_{xx}(0, t) = f_{xx}(1, t) = 0$  and  $\frac{\partial^4}{\partial x^4} f(\cdot, t) \in L^1(0, 1)$  hold, then there exists a unique solution of the considered problem (3.1.11)-(3.1.16).

## 3.2 On a nonlocal problem for the fractional order mixed PDE with singular coefficient

Studying non-local problems for mixed-type partial differential equations is one of the interesting target of investigations by many scientists because of their applications in physics, engineering. Non-local problems might develop when investigating various mathematical biology topics, also including soil moisture

prediction and plasma problems. It's worth noting that non-local conditions occur when modeling the flow around a profile by a subsonic velocity stream with a supersonic zone. We also remind that, in the case of mixed-type equations, when I. M. Gel'fand considered an example of gas motion in a channel surrounded by a porous medium, and the gas motion in a channel was described by a wave equation, while the diffusion equation was posed outside the channel, an interest in studying the problems for wave-diffusion equations arose [42]. Also, Ya. S. Uflyand investigated a problem [114] on the propagation of electric oscillations in compound lines when the losses on a semi-infinite line were neglected and the rest of the line was treated as a cable with no leaks and this problem reduced this problem to a mixed parabolic-hyperbolic type equation.

Considering above the problem formulated in Section 3.1 now we study non-local problem for mixed-type equation involving fractional wave equation generated by the right-hand sided bi-ordinal Hilfer fractional derivative and sub-diffusion equation with the regularized Caputo-like counterpart of hyper-Bessel fractional differential operator. We note that the similar problems have been studied for mostly left-hand sided fractional differential operators. Motivation of taking this operator is that we would like to consider the process which the time variable used negative called history.

### 3.2.1 Formulation of the problem

In this subsection, we investigate the following fractional order mixed differential equation involving the regularized Caputo-like counterpart of the hyper-Bessel operator and the bi-ordinal Hilfer derivative:

$$f(x, t) = \begin{cases} {}^C \left( t^\theta \frac{\partial}{\partial t} \right)^{\alpha_1} u(x, t) - \frac{1}{x} u_x(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_1, \\ D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t) - \frac{1}{x} u_x(x, t) - u_{xx}(x, t), & (x, t) \in \Omega_2, \end{cases} \quad (3.2.1)$$

in  $\Omega = \Omega_1 \cup \Omega_2 \cup Q$  domain, where  $\Omega_1 = \{(x, t) : 0 < x < 1, 0 < t < T\}$ ,  $\Omega_2 = \{(x, t) : 0 < x < 1, -T < t < 0\}$ ,  $Q = \{(x, t) : 0 < x < 1, t = 0\}$ ,  $0 < \alpha_1 \leq 1$ ,  $\theta < 1$ ,  $1 < \alpha_2, \beta_2 \leq 2$ ,  $0 \leq \mu \leq 1$ ,  $D_{0-}^{(\alpha_2, \beta_2)\mu}$  is the right-sided bi-ordinal Hilfer fractional derivative in the form (1.3.5) and  ${}^C\left(t^\theta \frac{d}{dt}\right)^{\alpha_1}$  is the regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator defined as (1.3.15).

**Problem.** Find a solution of eq.(3.2.1) in  $\Omega$ , which satisfies the following regularity conditions

$$\begin{aligned} u(x, t) &\in C(\overline{\Omega} \setminus Q), \quad u(\cdot, t) \in C^2(\Omega_1 \cup \Omega_2), \\ {}^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1} u(x, t) &\in C(\Omega_1), \quad D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t) \in C(\Omega_2) \end{aligned}$$

along with the boundary conditions

$$\lim_{x \rightarrow 0+} xu_x(x, t) = 0, \quad u(1, t) = 0, \quad -T \leq t \leq T, \quad (3.2.2)$$

non-local condition

$$\sum_{i=1}^m \zeta_i I_{0-}^{(1-\mu)(2-\beta_2)} u(x, \xi_i) = u(x, T), \quad 0 \leq x \leq 1, \quad (3.2.3)$$

and the gluing conditions

$$\lim_{t \rightarrow 0-} I_{0-}^{(1-\mu)(2-\beta_2)} u(x, t) = \lim_{t \rightarrow 0+} u(x, t), \quad 0 \leq x \leq 1, \quad (3.2.4)$$

$$\lim_{t \rightarrow 0-} \frac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u(x, t) = \lim_{t \rightarrow 0+} t^{1-(1-\theta)\alpha_1} u_t(x, t), \quad 0 < x < 1, \quad (3.2.5)$$

where  $-T \leq \xi_1 < \xi_2 < \dots < \xi_m < 0$ ,  $f(x, t)$  is a given function.

We notice that while we have been investigated the initial-boundary value or non-local problems involving popular class of differential operators like the Riemann-Liouville, Caputo, Hilfer fractional derivatives, Hadamard, Hilfer-Hadamard, Prabhakar, Atangana-Baleanu, the interest to another type of the differential operators is increased by many scientists, for instance, the

hyper-Bessel differential operator is becoming main target of research. The importance of the hyper-Bessel differential operator is increasing since introduced by Dimovski [23] because of its applications in science. For example, in [40] authors used hyper-Bessel differential operator to investigate heat diffusion equation for describing the Brownian motion. In [39] it is investigated fractional relaxation equation the regularized Caputo-like counterpart of the hyper-Bessel operator. Among these applications of fractional differential operators, it is worth to mention the work containing general idea of using those operators. For example, in [7] prof. Ashurov considered nonlocal problem in time for the general, self-adjoint operator for for subdiffusion equation.

Several local and nonlocal boundary value problems for mixed-type equations, ie, elliptic-hyperbolic and hyperbolic type equations were published [2],[118]. The interesting point is that the conjugation conditions are taken according to the considered mixed-type equations and domains [64], [13].

Before moving to investigation of main problem, first let us consider the Cauchy problem is investigated for the ordinary differential equation involving the right-sided bi-ordinal Hilfer fractional derivative:

$$\begin{cases} D_{0-}^{(\alpha,\beta)\mu} u(t) = \lambda u(t) + g(t), \\ \lim_{t \rightarrow 0-} I_{0-}^{2-\gamma} u(t) = \chi_0, \\ \lim_{t \rightarrow 0-} \frac{d}{dt} I_{0-}^{2-\gamma} u(t) = \chi_1, \end{cases} \quad (3.2.6)$$

where  $1 < \alpha, \beta \leq 2$ ,  $\gamma = \beta + \mu(2 - \beta)$ ,  $\chi_0, \chi_1 \in \mathbb{R}$ ,  $g(t)$  is the given function.

**Lemma 3.2.1.** *Let  $I_{0-}^\delta g(t) \in AC^2[-T, 0]$ . Then the solution of the problem (3.2.6) as follows:*

$$\begin{aligned} u(t) = & \chi_0(-t)^{\gamma-2} E_{\delta, \gamma-1}[\lambda(-t)^\delta] - \chi_1(-t)^{\gamma-1} E_{\delta, \gamma}[\lambda(-t)^\delta] + \\ & + \int_t^0 (z-t)^{\delta-1} E_{\delta, \delta}[\lambda(z-t)^\delta] g(z) dz, \end{aligned} \quad (3.2.7)$$

where  $\delta = \beta + \mu(\alpha - \beta)$ ,  $\gamma = \beta + \mu(2 - \beta)$ .

*Proof.* First, we rewrite the equation in 3.2.6 according to Remark 1.3.12 as follows:

$$I_{0-}^{\gamma-\delta} D_{0-}^{\gamma} u(t) = \lambda u(t) + g(t).$$

By applying  $I_{0-}^{\delta}$  operator to this equation we get

$$I_{0-}^{\gamma} D_{0-}^{\gamma} u(t) = \lambda I_{0-}^{\delta} u(t) + I_{0-}^{\delta} g(t)$$

and using the Property 1.3.5 of the Riemann-Liouville integral and differential operators yields

$$u(t) = \lambda I_{0-}^{\delta} u(t) + I_{0-}^{\delta} g(t) + \frac{(-t)^{\gamma-2} \xi_0}{\Gamma(\gamma-1)} - \frac{(-t)^{\gamma-1} \xi_1}{\Gamma(\gamma)}.$$

This integral equation has the following solution according to Lemma 1.3.7

$$u(t) = g^*(t) + \lambda \int_t^0 (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^{\delta}] g^*(s) ds = L_1(t) + L_2(t),$$

$$\text{where } g^*(t) = I_{0-}^{\delta} g(t) + \frac{(-t)^{\gamma-2} \chi_0}{\Gamma(\gamma-1)} - \frac{(-t)^{\gamma-1} \chi_1}{\Gamma(\gamma)}.$$

$$L_1(t) = \frac{(-t)^{\gamma-2} \chi_0}{\Gamma(\gamma-1)} - \frac{(-t)^{\gamma-1} \chi_1}{\Gamma(\gamma)} +$$

$$+ \lambda \int_t^0 (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^{\delta}] \left( \frac{(-s)^{\gamma-2} \chi_0}{\Gamma(\gamma-1)} - \frac{(-s)^{\gamma-1} \chi_1}{\Gamma(\gamma)} \right) ds,$$

$$L_2(t) = I_{0-}^{\delta} g(t) + \lambda \int_t^0 (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^{\delta}] I_{0-}^{\delta} g(s) ds.$$

If we use  $z = t - s$  substitution to the integral in  $L_1(t)$  and using the Lemma 1.4.4 and the Property (1.4.1) we can easily obtain the following result

$$L_1(t) = \chi_0 (-t)^{\gamma-2} E_{\delta,\gamma-1}[\lambda(-t)^{\delta}] - \chi_1 (-t)^{\gamma-1} E_{\delta,\gamma}[\lambda(-t)^{\delta}]. \quad (3.2.8)$$

Now we consider second integral on  $L_2(t)$

$$\begin{aligned}
 & \int_t^0 (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^\delta] I_{0-}^\delta f(s) ds = \\
 &= \frac{1}{\Gamma(\delta)} \int_t^0 (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^\delta] ds \int_s^0 (z-s)^{\delta-1} f(z) dz = \\
 &= \frac{1}{\Gamma(\delta)} \int_t^0 f(z) dz \int_t^z (z-s)^{\delta-1} (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^\delta] ds.
 \end{aligned}$$

By means of formula (1.4.1) we simplify the integrant as follows

$$\int_t^z (z-s)^{\delta-1} (s-t)^{\delta-1} E_{\delta,\delta}[\lambda(s-t)^\delta] ds = \Gamma(\delta) (z-t)^{2\delta-1} E_{\delta,2\delta}(\lambda(z-t)^\delta).$$

To clarify further, we use the Lemma 1.4.4, then the form of  $L_2(t)$  can be written as follows

$$L_2(t) = \int_t^0 (z-t)^{\delta-1} E_{\delta,\delta} [\lambda(z-t)^\delta] f(z) dz. \quad (3.2.9)$$

Finally, from (3.2.8) and (3.2.9), we can obtain the solution presented in Lemma 3.2.1. The similar lemma to Lemma 3.2.1 was also studied in [52] for equation with the left-sided Hilfer differential operator. The proof of Lemma 3.2.1 is completed.  $\square$

Now we recall some auxiliary results about parametric form of Bessel's equation, of order  $p$  given by

$$x^2 y'' + xy' + (\lambda^2 x^2 - p^2)y = 0 \quad (3.2.10)$$

and its the general solution which is bounded at near  $x = 0$

$$y(x) = C_1 J_p(\lambda x)$$



where  $J_p(\lambda x)$  is Bessel function of the first kind of order  $p$ . The value of  $\lambda$  denotes the zeros of the Bessel function  $J_p(x)$  when we impose the boundary condition  $y(1) = 0$ . In other words  $\lambda$  satisfies  $J_p(\lambda) = 0$  equation. We note that for arbitrary large  $k$  zeros of  $J_p(x)$  can be given by [110]

$$\lambda_k = \pi k + \frac{p\pi}{2} - \frac{\pi}{4} \quad (3.2.11)$$

Let  $\lambda_1, \lambda_2, \dots, \lambda_k, \dots$  be the positive roots of the equation  $J_p(x) = 0$  arranged in increasing order. The functions (see [110] page 221)

$$J_p(\lambda_1 x), J_p(\lambda_2 x), \dots, J_p(\lambda_k x), \dots \quad (3.2.12)$$

form an orthogonal system on  $[0, 1]$ , with weight  $x$ . For any function  $f(x)$  which is absolutely integrable on  $[0, 1]$  can be expanded into the Fourier series with respect to the system (3.2.12) i.e.,

$$f(x) = \sum_{k=1}^{\infty} c_k J_p(\lambda_k x)$$

where the constants

$$c_k = \frac{2}{J_{p+1}^2(\lambda_k)} \int_0^1 x f(x) J_p(\lambda_k x) dx, \quad k = 1, 2, \dots$$

are called Fourier-Bessel coefficients of  $f(x)$ .

**Theorem 3.2.2.** [110] *Let  $f(x)$  be a function defined on the interval  $[0, 1]$  such that  $f(x)$  is differentiable  $2s$  times ( $s \geq 1$ ) and*

- $f(0) = f'(0) = \dots = f^{(2s-1)}(0) = 0$
- $f^{(2s)}(x)$  is bounded (this derivative may not exist at certain points)
- $f(1) = f'(1) = \dots = f^{(2s-2)}(1) = 0$

*then the following inequalities satisfied by the Fourier-Bessel coefficients of  $f(x)$ :*

$$|c_k| \leq \frac{M}{\lambda_k^{2s-\frac{1}{2}}}.$$

**Theorem 3.2.3.** [110] *If  $p \geq 0$  and if*

$$|c_k| \leq \frac{M}{\lambda_k^{1+\varepsilon}}$$

*where  $\varepsilon$  is positive constant, then the series*

$$\sum_{k=1}^{\infty} c_k J_p(\lambda_k x)$$

*converges absolutely and uniformly on  $[0, 1]$ .*

### 3.2.2 Construction of a formal solution

Using the separation variables method we obtain the following spectral problem

$$X''(x) + \frac{1}{x}X'(x) + \lambda^2 X(x) = 0, \quad (3.2.13)$$

$$\lim_{x \rightarrow 0+} xX(x) = 0, \quad X(1) = 0, \quad (3.2.14)$$

It is clear that (3.2.13) is a Bessel equation of order zero; furthermore, the solution of the problem (3.2.13), (3.2.14) is a self-adjoint problem and its eigenfunctions are the Bessel functions given as follows

$$X_k(x) = J_0(\lambda_k x), \quad k = 1, 2, \dots, \quad (3.2.15)$$

and the eigenvalues  $\lambda_k$ , are the positive zeros of  $J_0(x)$ , i.e.,

$$\lambda_k = \pi k - \frac{\pi}{4} + \frac{\zeta(\lambda_k)}{k}.$$

The system of eigenfunctions  $\{X_k\}$  forms a complete orthogonal system in  $L^2(0, 1)$  (see [47], page 40), hence we can write sought function and given function in the form of series expansions as follows:

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) J_0(\lambda_k x), \quad (3.2.16)$$

$$f(x, t) = \sum_{k=1}^{\infty} f_k(t) J_0(\lambda_k x), \quad (3.2.17)$$

where  $u_k(t)$  is not known yet, and  $f_k(t)$  is the coefficient of Fourier-Bessel series, i.e,

$$f_k(t) = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x f(x, t) J_0(\lambda_k x) dx.$$

Let us introduce new notations:

$$\lim_{t \rightarrow 0-} I_{0-}^{(1-\mu)(2-\beta_2)} u(x, t) = \varphi(x), \quad 0 \leq x \leq 1, \quad (3.2.18)$$

$$\lim_{t \rightarrow 0-} \frac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u(x, t) = \psi(x), \quad 0 < x < 1, \quad (3.2.19)$$

$$\lim_{t \rightarrow 0+} u(x, t) = \tau(x), \quad 0 \leq x \leq 1, \quad (3.2.20)$$

here  $\varphi(x)$ ,  $\tau(x)$  and  $\psi(x)$  are unknown functions to be found later.

Further, after substituting (3.2.16) and (3.2.17) into the Eq.(3.2.1) and initial conditions (3.2.18), (3.2.19), (3.2.20), we obtain the following problems

$$\begin{cases} {}^C \left( t^\theta \frac{d}{dt} \right)^{\alpha_1} u_k(t) + \lambda_k^2 u_k(t) = f_k(t), \\ u_k(0+) = \tau_k, \end{cases} \quad (3.2.21)$$

and

$$\begin{cases} D_{0-}^{(\alpha_2, \beta_2)\mu} u_k(t) + \lambda_k^2 u_k(t) = f_k(t), \\ I_{0-}^{(1-\mu)(2-\beta_2)} u_k(0-) = \varphi_k, \\ \frac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u_k(0-) = \psi_k, \end{cases} \quad (3.2.22)$$

in  $\Omega_1$  and  $\Omega_2$  respectively.

The problem (3.2.21) was studied in [5] and by considering this result we can write the solution of (3.2.1) in  $\Omega_1$  which satisfies the conditions (3.2.2), (3.2.20) as

$$u(x, t) = \sum_{k=1}^{\infty} \left[ \tau_k E_{\alpha_1, 1} \left( -\frac{\lambda_k^2}{p^{\alpha_1}} t^{p\alpha_1} \right) + G_k(t) \right] J_0(\lambda_k x), \quad (3.2.23)$$

here  $p = 1 - \theta$  and

$$G_k(t) = \frac{1}{p^{\alpha_1} \Gamma(\alpha_1)} \int_0^t (t^p - \tau^p)^{\alpha_1-1} f_k(\tau) d(\tau^p) -$$

$$-\frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t (t^p - \tau^p)^{2\alpha_1-1} E_{\alpha_1, 2\alpha_1} \left[ -\frac{\lambda_k^2}{p^{\alpha_1}} (t^p - \tau^p)^{\alpha_1} \right] f_k(\tau) d(\tau^p),$$

where  $\tau_k$  is not known yet.

As we mentioned above the solution the problem (3.2.22) can be given by the formula (3.2.7) presented in Lemma 3.2.1 and by considering (3.2.16) we can write the solution of (3.2.1) which satisfies the conditions (3.2.2), (3.2.18), (3.2.19) in  $\Omega_2$  domain represented in the following form

$$\begin{aligned} u(x, t) = & \sum_{k=1}^{+\infty} \varphi_k(-t)^{\gamma_2-2} E_{\delta_2, \gamma_2-1} [-\lambda_k^2(-t)^{\delta_2}] J_0(\lambda_k x) - \\ & - \sum_{k=1}^{+\infty} \psi_k(-t)^{\gamma_2-1} E_{\delta_2, \gamma_2} [-\lambda_k^2(-t)^{\delta_2}] J_0(\lambda_k x) + \\ & + \sum_{k=1}^{+\infty} \int_t^0 (z-t)^{\delta_2-1} E_{\delta_2, \delta_2} [-\lambda_k^2(z-t)^{\delta_2}] f_k(z) dz J_0(\lambda_k x), \end{aligned} \quad (3.2.24)$$

where  $\gamma_2 = \beta_2 + \mu(2 - \beta_2)$ ,  $\delta_2 = \beta_2 + \mu(\alpha_2 - \beta_2)$  and  $\varphi_k$ ,  $\psi_k$  are not known yet.

After substituting (3.2.23) and (3.2.24) into gluing conditions with considering (3.2.16), (3.2.17) we obtain the following system of equations with respect to  $\tau_k$ ,  $\varphi_k$  and  $\psi_k$ :

$$\begin{cases} \psi_k = -\frac{\lambda_k^2}{\Gamma(\alpha_1)} \tau_k, \\ \tau_k = \varphi_k \end{cases} \quad (3.2.25)$$

With the help of non-local condition (3.2.3) and from (3.2.25), we find unknowns as follows

$$\tau_k = \varphi_k = \frac{F_k}{\Delta_k}, \quad (3.2.26)$$

$$\psi_k = \frac{-\lambda_k^2}{p^{\alpha_1} \Gamma(\alpha_1)} \frac{F_k}{\Delta_k}, \quad (3.2.27)$$

where

$$\Delta_k = \sum_{i=1}^m \zeta_i \left[ E_{\delta_2, 1}(-\lambda_k^2(-\xi_i)) + \frac{\lambda_k^2 \xi_i}{p^{\alpha_1} \Gamma(\alpha_1)} E_{\delta_2, 2}(-\lambda_k^2(-\xi_i)) \right] - E_{\alpha_1, 1} \left( -\frac{\lambda_k^2}{p^{\alpha_1}} T^{\alpha_1 p} \right),$$

$$F_k = G_k(T) - \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 (s - \xi_i)^{\delta_2 - \gamma_2 + 1} E_{\delta_2, \delta_2 - \gamma_2 + 2}(-\lambda_k^2 (s - \xi_i)^{\delta_2}) f_k(s) ds,$$

$$G_k(T) = \frac{1}{p^{\alpha_1} \Gamma(\alpha_1)} \int_0^T (T^p - \tau^p)^{\alpha_1 - 1} f_k(\tau) d(\tau^p) - \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^T (T^p - \tau^p)^{2\alpha_1 - 1} E_{\alpha_1, 2\alpha_1} \left[ -\frac{\lambda_k^2}{p^{\alpha_1}} (T^p - \tau^p)^{\alpha_1} \right] f_k(\tau) d(\tau^p).$$

If  $\Delta_k \neq 0$ , then we can find  $\tau_k$ ,  $\varphi_k$ ,  $\psi_k$  unknowns uniquely.

First we show that  $\Delta_k \neq 0$  for sufficiently large  $k$ . For this intention we use the Lemma 1.4.5 obtained from the properties of Wright-type function studied by A. Pskhu in [86].

By using the Lemma 1.4.5, we can calculate the behavior of  $\Delta_k$  at  $k \rightarrow \infty$ :

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Delta_k &= \lim_{|z_1| \rightarrow +\infty} \sum_{i=1}^m \zeta_i [E_{\delta_2, 1}(z_1) + \frac{1}{\Gamma(\alpha_1) p^{\alpha_1}} z_1 E_{\delta_2, 2}(z_1)] - \\ &- \lim_{|z_2| \rightarrow +\infty} E_{\alpha_1, 1}(z_2) = \sum_{i=1}^m \frac{\zeta_i}{\Gamma(\alpha_1) p^{\alpha_1} \Gamma(2 - \delta_2)}, \end{aligned}$$

where  $z_1 = -\lambda_k^2(-\xi_i)$ ,  $z_2 = -\frac{\lambda_k^2}{p^{\alpha_1}} T^{\alpha_1 p}$ ,  $\lambda_k = \pi k - \frac{\pi}{4}$ .

If  $\sum_{i=1}^m \frac{\zeta_i}{\Gamma(\alpha_1) p^{\alpha_1} \Gamma(2 - \delta_2)} > 0$  and from the last equality it is seen that  $\Delta_k > 0$  for sufficiently large  $k$ .

### 3.2.3 Uniqueness of the solution

In order to show the uniqueness of the solution, it is enough to prove that homogeneous problem has a trivial solution.

Let us first consider the following integral

$$u_k(t) = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x u(x, t) J_0(\lambda_k x) dx, \quad k = 1, 2, 3, \dots, \quad (3.2.28)$$

Then we introduce another function based on (3.2.28)

$$v_\varepsilon(t) = \frac{2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} xu(x, t) J_0(\lambda_k x) dx, \quad k = 1, 2, 3, \dots, \quad (3.2.29)$$

Applying  ${}^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1}$  and  $D_{0-}^{(\alpha_2, \beta_2)\mu}$  to (3.2.29) and using the equation (3.2.1) in homogeneous case with respect to  $t$  yield

$$\begin{aligned} {}^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1} v_\varepsilon(t) &= \frac{2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} {}^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1} u(x, t) x J_0(\lambda_k x) dx = \\ &= \frac{2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} \left[ u_{xx}(x, t) + \frac{1}{x} u_x(x, t) \right] x J_0(\lambda_k x) dx = \frac{-2\lambda_k^2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} u(x, t) x J_0(\lambda_k x) dx, \\ D_{0-}^{(\alpha_2, \beta_2)\mu} v_\varepsilon(t) &= \frac{2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t) x J_0(\lambda_k x) dx = \\ &= \frac{2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} \left[ u_{xx}(x, t) + \frac{1}{x} u_x(x, t) \right] x J_0(\lambda_k x) dx = \frac{-2\lambda_k^2}{J_1^2(\lambda_k)} \int_{\varepsilon}^{1-\varepsilon} u(x, t) x J_0(\lambda_k x) dx \end{aligned}$$

and integrating by parts twice the right sides of the equalities on  $t \in (0, T)$  and  $t \in (-T, 0)$ , respectively, and passing to the limit on  $\varepsilon \rightarrow +0$  yield

$$\begin{cases} {}^C\left(t^\theta \frac{d}{dt}\right)^{\alpha_1} u_k(t) + \lambda^2 u_k(t) = 0, & t > 0, \\ D_{0-}^{(\alpha_2, \beta_2)\mu} u_k(t) + \lambda^2 u_k(t) = 0, & t < 0. \end{cases} \quad (3.2.30)$$

Considering conditions in (3.2.21), (3.2.22) in homogeneous case, (3.2.30) has a solution  $u_k(t) = 0$  if  $\Delta_k \neq 0$ . Then from (3.2.28) and the completeness of the system  $X_k(x)$  in the space  $L^2(0, 1)$ ,  $u(x, t) \equiv 0$  in  $\overline{\Omega}$ . This completes the prove of uniqueness of the solution of the main problem.

### 3.2.4 Existence of the solution

In the below, we present the necessary conditions for given data to show the existence result by using well-known lemma about the upper bound of the Mittag-Leffler function and the theorem related to Fourier-Bessel series.

First, by considering above Lemma 1.4.1 and Theorem 3.2.2, we show the upper bound of  $G_k(t)$ :

$$\begin{aligned}
 |G_k(t)| &\leq \int_0^t |t^p - \tau^p|^{\alpha_1-1} |f_k(\tau)| d(\tau^p) + \\
 &+ \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t |t^p - \tau^p|^{2\alpha_1-1} |E_{\alpha_1, 2\alpha_1} \left[ -\frac{\lambda_k^2}{p^{\alpha_1}} (t^p - \tau^p)^{\alpha_1} \right]| |f_k(\tau)| d(\tau^p) \leq \\
 &\frac{1}{p^{\alpha_1} \Gamma(\alpha_1)} \int_0^t |t^p - \tau^p|^{\alpha_1-1} \frac{M}{\lambda_k^{7/2}} d(\tau^p) + \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t \frac{p^{\alpha_1} |t^p - \tau^p|^{2\alpha_1-1} M^*}{p^{\alpha_1} + \lambda_k^2 |t^p - \tau^p|^{\alpha_1}} \frac{M}{\lambda_k^{7/2}} d(\tau^p) \leq \\
 &\left[ \frac{M}{\lambda_k^{7/2} p^{\alpha_1} \Gamma(\alpha_1)} + \frac{M^* M}{p^{\alpha_1} \lambda_k^{7/2}} \right] \int_0^t |t^p - \tau^p|^{\alpha_1-1} d(\tau^p) \leq \\
 &\frac{1}{\lambda_k^{7/2}} \left[ \frac{M}{p^{\alpha_1} \Gamma(\alpha_1)} + \frac{M^* M}{p^{\alpha_1}} \right] \frac{|t|^{\alpha_1 p}}{\alpha_1} \leq \frac{M_1}{\lambda_k^{7/2}} |t|^{\alpha_1 p} \leq \frac{M_1}{\lambda_k^{7/2}} |T|^{\alpha_1 p},
 \end{aligned}$$

where

$$M_1 = \frac{M}{p^{\alpha_1} \Gamma(\alpha_1 + 1)} + \frac{M^* M}{\alpha_1 p^{\alpha_1}}.$$

By using the last inequality, Lemma 1.4.1 and Theorem 3.2.2, we can write the upper bound of  $F_k$ :

$$\begin{aligned}
 |F_k| &\leq |G_k(T)| + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{\delta_2 - \gamma_2 + 1} |E_{\delta_2, \delta_2 - \gamma_2 + 2}(-\lambda_k^2 (s - \xi_i)^{\delta_2})| |f_k(s)| ds \leq \\
 &\leq \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{\delta_2 - \gamma_2 + 1} \frac{M^*}{1 + \lambda_k^2 |s - \xi_i|^{\delta_2}} \frac{M}{\lambda_k^{7/2}} ds \leq
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{1-\gamma_2} \frac{MM^*}{\lambda_k^{11/2}} \leq \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \frac{(-\xi_i)^{2-\gamma_2}}{(2-\gamma_2)} \frac{MM^*}{\lambda_k^{11/2}} \leq, \\
 &\leq \frac{M_2}{\lambda_k^{7/2}}, \quad M_2 = M_1 T^{\alpha_1 p} + \sum_{i=1}^m \frac{\zeta_i MM^* (-\xi_i)^{2-\gamma_2}}{(2-\gamma_2) \lambda_k^2},
 \end{aligned}$$

or

$$|F_k| \leq \frac{M_2}{\lambda_k^{7/2}}, \quad (3.2.31)$$

Now considering (3.2.31) and assuming  $\Delta_k \neq 0$  then, we write upper bounds of  $\tau_k$ ,  $\varphi_k$ ,  $\psi_k$  in (3.2.26) and (3.2.27).

$$|\tau_k| = |\varphi_k| \leq \left| \frac{1}{\Delta_k} \right| |F_k| \leq \frac{M_2}{|\Delta_k| \lambda_k^{7/2}}, \quad (3.2.32)$$

$$|\psi_k| = \left| \frac{-\lambda_k^2}{p^{\alpha_1} \Gamma(\alpha_1)} \right| \left| \frac{F_k}{\Delta_k} \right| \leq \frac{M_2}{p^{\alpha_1} \Gamma(\alpha_1) |\Delta_k| \lambda_k^{3/2}}. \quad (3.2.33)$$

For proving the existence of the solution, we need to show uniform convergence of series representations of  $u(x, t)$ ,  $u_x(x, t)$ ,  $u_{xx}(x, t)$ ,  ${}^C(t^\theta \frac{\partial}{\partial t})^\alpha u(x, t)$  and  $D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t)$  by using the solutions (3.2.23) and (3.2.24) in  $\Omega_1$  and  $\Omega_2$  respectively.

According to the last inequality (3.2.32) and Theorem 3.2.2, then we can present the existence of the solutions in both domains.

$$\begin{aligned}
 |u(x, t)| &\leq \sum_{k=1}^{\infty} |u_k(t)| |J_0(\lambda_k x)| \leq \sum_{k=1}^{\infty} |u_k(t)| \leq \\
 &\sum_{k=1}^{\infty} \left[ |\tau_k| \left| E_{\alpha_1, 1} \left( -\frac{\lambda_k^2}{p^{\alpha_1}} t^{\alpha_1 p} \right) \right| + |G_k(t)| \right] \leq \\
 &\leq \sum_{k=1}^{\infty} \left( \frac{p^{\alpha_1}}{p^{\alpha_1} + \lambda_k^2 |t^{p\alpha_1}|} \frac{M_2}{|\Delta_k| \lambda_k^{7/2}} + \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} \right).
 \end{aligned}$$

One can shows that the series representation of  $u(x, t)$  is bounded by convergent numerical series and by Weierstrass M-test, the series of  $u(x, t)$  converges uniformly in  $\Omega_1$ .



Now we remind some properties of Bessel functions [110]:

$$J'_0(x) = -J_1(x); \quad 2J'_1(x) = J_0(x) - J_2(x)$$

and asymptotic formula  $|J_p(\lambda_k x)| \leq \frac{2A}{\sqrt{\lambda_k x}}$ ,  $p > -1/2$ ,  $A = \text{const}$ .

It is not difficult to see that the series representation of  $u_{xx}(x, t)$  is bigger than  $u_x(x, t)$  hence it is enough to show the uniform convergence of  $u_{xx}(x, t)$ .

By using these properties we have

$$|u_{xx}(x, t)| \leq \sum_{k=1}^{\infty} |u_k(t)| \left| \frac{d^2}{dx^2} J_0(\lambda_k x) \right| = \sum_{k=1}^{\infty} |u_k(t)| \frac{\lambda_k^2}{2} |J_2(\lambda_k x) - J_0(\lambda_k x)|$$

And as a similar way of  $u(x, t)$  we can show that

$$|u_{xx}(x, t)| \leq \sum_{k=1}^{\infty} \left( \frac{p^{\alpha_1}}{p^{\alpha_1} + \lambda_k^2 |t^{p\alpha_1}|} \frac{M_2}{|\Delta_k| \lambda_k^{7/2}} + \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} \right) \frac{2A}{\sqrt{\lambda_k x}}.$$

From the last inequality we can see that the series representation of  $u_{xx}(x, t)$  is bounded by convergent series. According to Weierstrass M-test, the series of  $u_{xx}(x, t)$  converges uniformly in  $\Omega_1$ .

The uniform convergence of  ${}^C (t^\theta \frac{\partial}{\partial t})^\alpha u(x, t)$  which is defined as

$${}^C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) = u_{xx}(x, t) - \frac{1}{x} u_x(x, t) + f(x, t)$$

is similar to the way of showing convergence of the series representation of  $u_{xx}(x, t)$ .

Most fundamental result of mixed type equation presented by Chaplygin which is closely connected with the theory of gas flow.

In  $\Omega_2$  domain it is enough to show the uniform convergence of  $u_{xx}(x, t)$  which is bigger than other series. Hence the convergence of the series of  $u(x, t)$ ,  $u_x(x, t)$ ,  $D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t)$  can be derived from the uniform convergence of  $u_{xx}(x, t)$ .

From Theorem 3.2.2 and Lemma 1.4.1, in  $\Omega_2$  we can have

$$\begin{aligned}
 |u_{xx}(x, t)| &\leq \sum_{k=1}^{\infty} |u_k(t)| \frac{\lambda_k^2}{2} \left| J_2(\lambda_k x) - J_0(\lambda_k x) \right| \leq \\
 &\sum_{k=1}^{\infty} \lambda_k^2 |\varphi_k| |(-t)^{\gamma_2-2}| |E_{\delta_2, \gamma_2-1}(-\lambda_k^2(-t)^{\delta_2})| + \\
 &+ \sum_{k=1}^{\infty} \lambda_k^2 |\psi_k| |(-t)^{\gamma_2-1}| |E_{\delta_2, \gamma_2}(-\lambda_k^2(-t)^{\delta_2})| + \\
 &\sum_{k=1}^{+\infty} \lambda_k^2 \int_t^0 |z-t|^{\delta_2-1} |E_{\delta_2, \delta_2}[-\lambda_k^2(z-t)^{\delta_2}]| |f_k(z)| dz \leq \\
 &\leq \sum_{k=1}^{\infty} \left[ \frac{\lambda_k^2 M_2}{|\Delta_k| \lambda_k^{7/2}} \frac{|(-t)^{\gamma_2-2}| M^*}{1 + \lambda_k^2 |(-t)^{\delta_2}|} + \frac{\lambda_k^2 M_2}{|\Delta_k| \lambda_k^{3/2} p^{\alpha_1} \Gamma(\alpha_1)} \frac{|(-t)^{\gamma_2-1}| M^*}{1 + \lambda_k^2 |(-t)^{\delta_2}|} \right] + \\
 &+ \sum_{k=1}^{+\infty} \lambda_k^2 \int_t^0 |z-t|^{\delta_2-1} \frac{M^*}{1 + \lambda_k^2 |(z-t)^{\delta_2}|} \frac{M}{\lambda_k^{7/2}} dz \leq \\
 &\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{3/2}} \left[ \frac{M_2 M^* T^{\gamma_2-2}}{|\Delta_k| (1 + \lambda_k^2 T^{\delta_2})} + \frac{M_2 M^* T^{\gamma_2-\delta_2-1}}{|\Delta_k|} + \right. \\
 &\quad \left. + \frac{M M^* \ln(1 + \lambda_k^2 T^{\delta_2})}{\delta_2 \lambda_k^2} \right] \leq \sum_{k=1}^{\infty} \frac{M_3}{\lambda_k^{3/2}}
 \end{aligned}$$

where  $\lim_{\lambda_k \rightarrow \infty} \frac{\ln(1 + \lambda_k^2 T^{\delta_2})}{\delta_2 \lambda_k^2} < \infty$  according to l'Hopital's rule. It can be seen that the series representation of  $u_{xx}(x, t)$  is bounded by convergent numerical series and due to Weierstrass M-test, the series of  $u_{xx}(x, t)$  converges uniformly in  $\Omega_2$

Using (3.2.32), (3.2.33) and Theorem 3.2.2, Lemma 1.4.1, we can show the uniform convergence of  $u(x, t)$ ,  $u_x(x, t)$ , and  $D_{0-}^{(\alpha_2, \beta_2)\mu} u(x, t)$  in a similar method used for  $u_{xx}(x, t)$  in  $\Omega_2$ .

Finally, we have proved the uniqueness and existence of the solution to the considered problem as stated in the following theorem.

**Theorem 3.2.4.** *Let  $\Delta_k \neq 0$  and  $\sum_{i=1}^m \frac{\zeta_i}{\Gamma(\alpha_1) p^{\alpha_1} \Gamma(2 - \delta_2)} > 0$ , and also the*

following conditions hold for  $I_{0-}^{\delta} f(x, \cdot) \in AC^2[-T, 0-]$  and  $f(x, \cdot) \in C_{\mu}[0+, T]$  such that

- $f(0, t) = f'_x(0, t) = \dots = f'''_x(0, t) = 0;$
- $f(1, t) = f'_x(1, t) = f''_x(1, t) = 0;$
- $\frac{\partial^4}{\partial x^4} f(x, t)$  is bounded;

then, there exist the unique solution of the considered problem which is represented by

$$u(x, t) = \sum_{k=1}^{\infty} U_k(t) J_0(\lambda_k x)$$

where

$$U_k(t) = \begin{cases} \tau_k E_{\alpha_1, 1} \left( -\frac{\lambda_k^2}{p^{\alpha_1}} t^{p\alpha_1} \right) + G_k(t), & t > 0, \\ \varphi_k(-t)^{\gamma_2-2} E_{\delta_2, \gamma_2-1} [-\lambda_k^2 (-t)^{\delta_2}] - \psi_k(-t)^{\gamma_2-1} E_{\delta_2, \gamma_2} [-\lambda_k^2 (-t)^{\delta_2}] + \\ \quad + \int_t^0 (z-t)^{\delta_2-1} E_{\delta_2, \delta_2} [-\lambda_k^2 (z-t)^{\delta_2}] f_k(z) dz, & t < 0, \end{cases}$$

here  $p = 1 - \theta$  and

$$G_k(t) = \frac{1}{p^{\alpha_1} \Gamma(\alpha_1)} \int_0^t (t^p - \tau^p)^{\alpha_1-1} f_k(\tau) d(\tau^p) - \\ - \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t (t^p - \tau^p)^{2\alpha_1-1} E_{\alpha_1, 2\alpha_1} \left[ -\frac{\lambda_k^2}{p^{\alpha_1}} (t^p - \tau^p)^{\alpha_1} \right] f_k(\tau) d(\tau^p).$$

### 3.3 On the nonlocal problem in time for mixed PDE involving time fractional wave and subdiffusion equations

In this section we analyze the hybrid nonlocal boundary-value problem made by combining two different elements of the problems considered in the sections

3.1 and 3.2. Detailed information about this investigation has been published [121] in the journal of *Uzbek Mathematical Journal*.

In the present work, we consider the following mixed PDE with fractional subdiffusion and fractional wave equation in both parts of the domain involving bi-ordinal Hilfer derivative:

$$f(x, t) = \begin{cases} L_1 u \equiv D_{0+}^{(\alpha_1, \beta_1)\mu_1} u(x, t) - u_{xx}(x, t), & t > 0 \\ L_2 u \equiv D_{0-}^{(\alpha_2, \beta_2)\mu_2} u(x, t) - u_{xx}(x, t), & t < 0 \end{cases} \quad (3.3.1)$$

in mixed domain  $\Omega = \Omega_1 \cup \Omega_2 \cup AB$ .

Where  $f(x, t)$  is a given function,  $i - 1 < \alpha_i, \beta_i < i$ ,  $0 \leq \mu_i \leq 1$ ,  $i = \overline{1, 2}$ ,  $\Omega_1 = \{(x, t) : 0 < x < l, 0 < t < T\}$ ,  $\Omega_2 = \{(x, t) : 0 < x < l, -T < t < 0\}$ ,  $T > 0$ ,  $AB = \{(x, t) : 0 < x < l, t = 0\}$ ,

Nonlocal BVP for Eq.(3.3.1) in  $\Omega$  can be formulated as follows:

**Problem B.** Find a solution  $u(x, t)$  of equation (3.3.1) which is subject to the following regularity conditions

$$t^{1-\gamma_1} u(x, t), \quad t^{1-\gamma_1} D_{0+}^{(\alpha_1, \beta_1)\mu_1} u(x, t) \in C(\overline{\Omega_1}), \quad (-t)^{2-\gamma_2} u(x, t) \in C(\overline{\Omega_2}),$$

$$(-t)^{2-\gamma_2} D_{0-}^{(\alpha_2, \beta_2)\mu_2} u(x, t) \in C(\overline{\Omega_2}), \quad u_{xx} \in C(\Omega_1 \cup \Omega_2),$$

submitted to the boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \in [-T, 0) \cap (0, T] \quad (3.3.2)$$

and non-local condition

$$u(x, -T) = u(x, T) + \psi(x), \quad 0 \leq x \leq l, \quad (3.3.3)$$

and also it satisfies the conjugation conditions on  $AB$

$$\lim_{t \rightarrow +0} I_{0+}^{1-\gamma_1} u(x, t) = \lim_{t \rightarrow -0} I_{0-}^{2-\gamma_2} u(x, t), \quad 0 \leq x \leq l, \quad (3.3.4)$$

$$\lim_{t \rightarrow +0} t^{1-\delta_1} \left( \frac{\partial}{\partial t} I_{0+}^{1-\gamma_1} u(x, t) \right) = \lim_{t \rightarrow -0} \frac{\partial}{\partial t} I_{0-}^{1-\gamma_2} u(x, t) \quad 0 < x < l. \quad (3.3.5)$$

where  $\gamma_i = \beta_i + \mu_i(i - \beta_i)$   $\delta_i = \beta_i + \mu_i(\alpha_i - \beta_i)$ ,  $(i = \overline{1, 2})$ ,  $\psi(x)$  is a given function such that  $\psi(0) = \psi(l) = 0$ .

We note works of A. Pskhu [88], [89] where main boundary value problems for diffusion-wave equation with the Riemann-Liouville fractional derivative were investigated by the method of Green's functions.

### 3.3.1 Investigation of Problem

First let us introduce the following new notations

$$\tau(x) = \lim_{t \rightarrow +0} I_{0+}^{1-\gamma_1} u(x, t), \quad 0 \leq x \leq l, \quad (3.3.6)$$

$$\varphi(x) = \lim_{t \rightarrow -0} I_{0-}^{2-\gamma_2} u(x, t), \quad 0 \leq x \leq l, \quad (3.3.7)$$

$$\nu(x) = \lim_{t \rightarrow -0} \frac{\partial}{\partial t} I_{0-}^{2-\gamma_2} u(x, t), \quad 0 < x < l. \quad (3.3.8)$$

For solving the problem we use the method of separation of variables for homogeneous equation corresponding (3.3.1) along with the conditions (3.3.2) and we obtain the same spectral problem as given in the section 3.1 whose its eigenvalues and eigenfunctions are in the following forms

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin(\sqrt{\lambda_n}x), \quad n = 1, 2, 3, \dots \quad (3.3.9)$$

The system of  $X_n(x)$  in the form (3.3.9) is the orthogonal basis in  $L_2(0, l)$  [79], for that reason we can represent the solution  $u(x, t)$  and the given function  $f(x, t)$  in the form of series expansions as follows

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(\sqrt{\lambda_n}x) \quad (3.3.10)$$

and

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(\sqrt{\lambda_n}x), \quad (3.3.11)$$

where

$$f_n(t) = \frac{2}{l} \begin{cases} \int_0^t f(x, t) \sin(\sqrt{\lambda_n} x) dx, & t > 0, \\ \int_t^0 f(x, t) \sin(\sqrt{\lambda_n} x) dx, & t < 0. \end{cases} \quad (3.3.12)$$

Substituting (3.3.10) and (3.3.11) into the equation (3.3.1) along with the conditions (3.3.6), (3.3.7) and (3.3.8) we obtain the problem for the ordinary fractional differential equations in  $\Omega_1$  and  $\Omega_2$  respectively.

The ordinary fractional differential equation with respect to  $t$  corresponding Eq.(3.3.1) has been studied in [18] for  $t > 0$ . Hence, we can write the solution of the Eq.(3.3.1) in  $\Omega_1$  which satisfies conditions (3.3.2), (3.3.6) as follows:

$$u(x, t) = \sum_{n=1}^{+\infty} \left[ \tau_n t^{\gamma_1-1} E_{\delta_1, \gamma_1}(-\lambda_n t^{\delta_1}) + \int_0^t (t-s)^{\delta_1-1} E_{\delta_1, \delta_1}[-\lambda_n(t-s)^{\delta_1}] f_n(s) ds \right] \sin(\sqrt{\lambda_n} x), \quad (3.3.13)$$

where  $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ .

Using representations (3.3.13), we evaluate  $t^{1-\delta_1} \left( I_{0+}^{1-\gamma_1} u(x, t) \right)_t$ :

$$\begin{aligned} t^{1-\delta_1} \left( I_{0+}^{1-\gamma_1} u(x, t) \right)_t &= t^{1-\delta_1} \sum_{n=1}^{\infty} \left[ \frac{d}{dt} \tau_n E_{\delta_1, 1}(-\lambda_n t^{\delta_1}) + \right. \\ &\quad \left. + \frac{d}{dt} \int_0^t (t-s)^{2\delta_1-1} E_{\delta_1, \delta_1-\gamma_1+1}[-\lambda_n(t-s)^{\delta_1}] f_n(s) ds \right] \sin(\sqrt{\lambda_n} x) = \\ &= t^{1-\delta_1} \sum_{n=1}^{\infty} \left( -\lambda_n \tau_n t^{\delta_1-1} E_{\delta_1, 1}(-\lambda_n t^{\delta_1}) - f(0) t^{2\delta_1-1} E_{\delta_1, \delta_1-\gamma_1+1}(-\lambda t^{\delta_1}) \right) \sin(\sqrt{\lambda_n} x) - \\ &\quad - t^{1-\delta_1} \sum_{n=1}^{\infty} \left( \int_0^t (t-s)^{2\delta_1-1} E_{\delta_1, \delta_1-\gamma_1+1}[-\lambda_n(t-s)^{\delta_1}] f'_n(s) ds \right) \sin(\sqrt{\lambda_n} x). \end{aligned}$$

According to the above evaluation, we can calculate the limit

$$\lim_{t \rightarrow +0} t^{1-\delta_1} \left( I_{0+}^{1-\gamma} u(x, t) \right)_t = \sum_{n=1}^{\infty} (-\lambda_n) \tau_n \sin(\sqrt{\lambda_n} x), \quad 0 < x < l. \quad (3.3.14)$$

Considering notations (3.3.6), (3.3.7) and conjugation condition (3.3.4), as such from (3.3.8), (3.3.14) and conjugation condition (3.3.5), we obtain the following linear equations:

$$\begin{cases} \tau_n = \varphi_n, \\ -\frac{\lambda_n \tau_n}{\Gamma(\delta_1)} = \nu_n, \end{cases} \quad (3.3.15)$$

where  $\tau_n$ ,  $\varphi_n$  and  $\nu_n$  are Fourier coefficients of the unknown functions  $\tau(x)$ ,  $\varphi(x)$  and  $\nu(x)$  respectively.

Now we will establish another functional relation which is determine from (3.3.3). For this aim, we need the solution of the problem intended to solve  $L_2 u = 0$  equation with the conditions (3.3.7), (3.3.8). After applying method of separation variables, we have spectral problem which its eigenvalues and eigenfunctions as given in (3.3.9) and the problem (3.2.22) considered in the section 3.2 for ordinary differential equation involving the right-sided bi-ordinal Hilfer fractional differential operator.

According to Lemma 3.2.1 and (3.3.9) we can write the solution of  $L_2 u = 0$  satisfying (3.3.2), (3.3.7), (3.3.8) conditions as follows:

$$u(x, t) = \sum_{n=1}^{+\infty} \left[ \varphi_n(-t)^{\gamma_2-2} E_{\delta_2, \gamma_2-1}[-\lambda_n(-t)^{\delta_2}] - \nu_n(-t)^{\gamma_2-1} E_{\delta_2, \gamma_2}[-\lambda_n(-t)^{\delta_2}] + \int_t^0 (z-t)^{\delta_2-1} E_{\delta_2, \delta_2}[-\lambda_n(z-t)^{\delta_2}] f_n(z) dz \right] \sin(\sqrt{\lambda_n} x) \quad (3.3.16)$$

By considering (3.3.10) and (3.3.11) we substite (3.3.13) and (3.3.16) into

(3.3.3) we deduce that

$$\begin{aligned} \psi_n = & \varphi_n T^{\gamma_2-2} E_{\delta_2, \gamma_2-1}(-\lambda_n T^{\delta_2}) - \nu_n T^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_n T^{\delta_2}) + \\ & + \int_{-T}^0 (z+T)^{\delta_2-1} E_{\delta_2, \delta_2}(-\lambda(z+T)^{\delta_2}) f_n(z) dz - \tau_n T^{\gamma_1-1} E_{\delta_1, \gamma_1}(-\lambda T^{\delta_1}) \\ & - \int_0^T (T-z)^{\delta_1-1} E_{\delta_1, \delta_1}(-\lambda_n(T-z)^{\delta_1}) f_n(z) dz, \end{aligned} \quad (3.3.17)$$

where  $\varphi_n$  is a Fourier coefficient of  $\varphi(x)$ , i.e,

$$\psi_n = \frac{2}{l} \int_0^l \psi(x) \sin(\sqrt{\lambda_n} x) dx$$

From the system of equations (3.3.15), (3.3.17), one can determine  $\tau_n, \varphi_n, \nu_n$  unknowns in the following forms

$$\tau_n = \frac{1}{\Delta_n} (\psi_n + F_n), \quad (3.3.18)$$

$$\nu_n = \frac{-\lambda_n}{\Delta_n} (\psi_n + F_n), \quad (3.3.19)$$

$$\varphi_n = \frac{1}{\Delta_n} (\psi_n + F_n), \quad (3.3.20)$$

where

$$\begin{aligned} \Delta_n = & T^{\gamma_2-2} E_{\delta_2, \gamma_2-1}(-\lambda_n T^{\delta_2}) + \\ & + \frac{\lambda_n T^{\gamma_2-1}}{\Gamma(\delta_1)} E_{\delta_2, \gamma_2}(-\lambda_n T^{\delta_2}) - T^{\gamma_1-1} E_{\delta_1, \gamma_1}(-\lambda T^{\delta_1}), \end{aligned} \quad (3.3.21)$$

$$\begin{aligned} F_n = & \int_0^T (T-z)^{\delta_1-1} E_{\delta_1, \delta_1}(-\lambda_n(T-z)^{\delta_1}) f_n(z) dz - \\ & - \int_{-T}^0 (z+T)^{\delta_2-1} E_{\delta_2, \delta_2}(-\lambda_n(z+T)^{\delta_2}) f_n(z) dz. \end{aligned}$$



### 3.3.2 The uniqueness of solution

We assume that there exist two different  $u_1(x, t)$  and  $u_2(x, t)$  solutions of the main problem. Then it is enough to show that  $u(x, t) = u_1(x, t) - u_2(x, t)$  is a trivial solution of the homogeneous problem.

Let  $u(x, t)$  be a solution of the homogeneous problem.

Let us first consider the following integral

$$u_n(t) = \int_0^1 u(x, t) \sin(\sqrt{\lambda_n}x) dx, \quad n = 1, 2, 3, \dots, \quad (3.3.22)$$

Then we introduce another function based on (3.3.22)

$$v_n^\varepsilon(t) = \int_\varepsilon^{1-\varepsilon} u(x, t) \sin(\sqrt{\lambda_n}x) dx, \quad n = 1, 2, 3, \dots, \quad (3.3.23)$$

Applying  $D_{0+}^{(\alpha_1, \beta_1)\mu_1}$  and  $D_{0-}^{(\alpha_2, \beta_2)\mu_2}$  to (3.3.23) and using the equation (3.3.1)

$$\begin{aligned} D_{0+}^{(\alpha_1, \beta_1)\mu_1} v_\varepsilon(t) &= -2 \int_\varepsilon^{1-\varepsilon} D_{0+}^{(\alpha_1, \beta_1)\mu_1} u(x, t) \sin(\sqrt{\lambda_n}x) dx = \\ &= -2 \int_\varepsilon^{1-\varepsilon} u_{xx}(x, t) \sin(\sqrt{\lambda_n}x) dx \\ D_{0-}^{(\alpha_2, \beta_2)\mu_2} v_\varepsilon(t) &= -2 \int_\varepsilon^{1-\varepsilon} D_{0-}^{(\alpha_2, \beta_2)\mu_2} u(x, t) \sin(\sqrt{\lambda_n}x) dx = \\ &= -2 \int_\varepsilon^{1-\varepsilon} u_{xx}(x, t) \sin(\sqrt{\lambda_n}x) dx \end{aligned}$$

and integrating by parts twice the right sides of the equalities on  $t \in (0, T)$  and  $t \in (-T, 0)$ , respectively, and passing to the limit on  $\varepsilon \rightarrow +0$  yield

$$\begin{cases} D_{0+}^{(\alpha_1, \beta_1)\mu_1} u_n(t) + \lambda^2 u_n(t) = 0, & t > 0, \\ D_{0-}^{(\alpha_2, \beta_2)\mu_2} u_n(t) + \lambda^2 u_n(t) = 0, & t < 0. \end{cases} \quad (3.3.24)$$

Considering conditions (3.3.6), (3.3.7), (3.3.8) in homogeneous case, (3.3.24) has a solution  $u_n(t) = 0$  if  $\Delta_n \neq 0$  in (3.3.21). Then from (3.3.22) and the completeness of the system  $\{X_n(x)\}$  in the space  $L^2(0, l)$ ,  $u(x, t) \equiv 0$  in  $\bar{\Omega}$ . This completes the prove of uniqueness of the solution of Problem B.

### 3.3.3 The existence of solution

First of all, we prove that  $\Delta_n \neq 0$  for sufficiently large  $n$ . Considering Lemma 1.4.5 we can show

$$\begin{aligned} \lim_{n \rightarrow +\infty} \Delta_n &= \lim_{\lambda_n \rightarrow +\infty} \Delta_n = \lim_{|z_1| \rightarrow +\infty} \left( T^{\gamma_2-2} E_{\delta_2, \gamma_2-1}(z_1) - \frac{T^{\gamma_2-1-\delta_2}}{\Gamma(\delta_1)} E_{\delta_2, \gamma_2}(z_1) \right) - \\ &\quad - \lim_{|z_2| \rightarrow +\infty} T^{\gamma_1-1} E_{\delta_1, \gamma_1}(z_2) = \frac{T^{\gamma_2-\delta_2-1}}{\Gamma(\delta_1)\Gamma(\gamma_2-\delta_2)} > 0. \end{aligned}$$

In other words, it confirms that  $\Delta_n > 0$  for any sufficiently large  $n$ .

For showing the existence of the result, we prove the uniform convergence of the series of  $u(x, t)$ ,  $u_{xx}(x, t)$  and  $D_{0\pm}^{(\alpha_i, \beta_i)\mu_i} u(x, t)$ ,  $i = \overline{1, 2}$ .

Second of all, we get the estimates of the function  $u(x, t)$  in  $\Omega_1$  with the help of Lemma 1.4.1:

$$\begin{aligned} |t^{1-\gamma_1} u(x, t)| &\leq \sum_{n=1}^{\infty} |\tau_n| E_{\delta_1, \gamma_1}(-\lambda_n t^{\delta_1})| + \\ &\quad + \sum_{n=1}^{\infty} t^{1-\gamma_1} \int_0^t |t-s|^{\delta_1-1} |E_{\delta, \delta_1}[-\lambda_n(t-s)^{\delta_1}]| |f_n(s)| ds = \\ &= \sum_{n=1}^{\infty} |\tau_n| E_{\delta_1, \gamma_1}(-\lambda_n t^{\delta_1})| + |f_n(0)| |t|^{\delta_1} |E_{\delta_1, \delta_1+1}(-\lambda_n t^{\delta_1})| + \\ &\quad + \sum_{n=1}^{\infty} |t|^{1-\gamma_1} \int_0^t |t-s|^{\delta_1} |E_{\delta, \delta_1+1}[-\lambda_n(t-s)^{\delta_1}]| |f'_n(s)| ds = \\ &\leq \sum_{n=1}^{\infty} \left( \frac{|\tau_n| |M|}{1 + \lambda_n |t^{\delta_1}|} + \frac{|f_n(0)| |t|^{\delta_1+1-\gamma_1}}{1 + \lambda_n |t|^{\delta_1}} + T^{1-\gamma_1} \int_0^t \frac{|t-s|^{\delta_1} M}{1 + \lambda_n |t-s|^{\delta_1}} |f'_n(s)| ds \right). \end{aligned}$$

If  $\tau(x) \in C[0, l]$ ,  $\tau'(x) \in L^2(0, l)$  and  $f(x, t) \in C^{0,1}[0, l] \times [0, T]$ , then the series of  $u(x, t)$  is bounded with convergent numerical series. Note that the condition  $\tau'(x) \in L^2(0, l)$  is required for  $t \rightarrow +0$ . From Weierstrass M-test the series of  $u(x, t)$  is considered uniformly convergent in  $\Omega_1$ .

As such for estimate  $u(x, t)$  in  $\Omega_2$  after integrating by parts

$$\begin{aligned}
 |(-t)^{2-\gamma_2} u(x, t)| &\leq \sum_{n=1}^{+\infty} \left( \varphi_n \left| E_{\delta_2, \gamma_2-1}[-\lambda_n(-t)^{\delta_2}] \right| + \nu_n \left| -t \left| E_{\delta_2, \gamma_2}[-\lambda_n(-t)^{\delta_2}] \right| \right| \right) + \\
 &\quad + \sum_{n=1}^{\infty} \left| -t \right|^{2-\gamma_2+\delta_2} |f_n(0)| \left| E_{\delta_2, \delta_2+1}(-\lambda_n(-t)^{\delta_2}) \right| + \\
 &\quad + \sum_{n=1}^{\infty} \left| -t \right|^{2-\gamma_2} \int_t^0 |z-t|^{\delta_2} \left| E_{\delta_2, \delta_2+1}[-\lambda_n(z-t)^{\delta_2}] \right| |f'_n(z)| dz = \\
 &\leq \sum_{n=1}^{\infty} \left( \frac{\varphi_n M}{1 + \lambda_n | -t |^{\delta_2}} + \frac{\nu_n | -t | M}{1 + \lambda_n | -t |^{\delta_2}} \right) + \\
 &\quad + \sum_{n=1}^{\infty} \left( \frac{|f_n(0)| | -t |^{\delta_2} M}{1 + \lambda_n | -t |^{\delta_2}} + T^{2-\gamma_2} \int_t^0 \frac{|z-t|^{\delta_2} M}{1 + \lambda_n |z-t|^{\delta_2}} |f'_n(z)| dz \right).
 \end{aligned}$$

If  $\varphi(x), \nu(x) \in C[0, l]$ ,  $\varphi(x), \nu(x) \in L^2(0, l)$  and  $f(x, t) \in C^{0,1}[0, l] \times [-T, 0]$ , then the series of  $u(x, t)$  is bounded with convergent numerical series with respect to  $n$  and from Weierstrass M-test the series of  $u(x, t)$  converges uniformly in  $\Omega_2$ .

Next, we show the uniform convergence of the series representation of  $u_{xx}(x, t)$ , which is given by in  $\Omega_1$

$$\begin{aligned}
 u_{xx}(x, t) &= - \sum_{n=1}^{\infty} \lambda_n \tau_n t^{\gamma_1-1} E_{\delta_1, \gamma_1}(-\lambda_n t^{\delta_1}) - \\
 &\quad - \sum_{n=1}^{\infty} \lambda_n \int_0^t (t-s)^{\delta_1-1} E_{\delta_1, \delta_1}[-\lambda_n(t-s)^{\delta_1}] f_n(s) ds \sin(\sqrt{\lambda_n} x),
 \end{aligned}$$

and as such given in  $\Omega_2$

$$\begin{aligned} u_{xx}(x, t) = & - \sum_{n=1}^{\infty} \lambda_n \varphi_n(-t)^{\gamma_2-2} E_{\delta_2, \gamma_2-1}[-\lambda_n(-t)^{\delta_2}] \sin(\sqrt{\lambda_n}x) + \\ & + \sum_{n=1}^{\infty} \lambda_n \nu_n(-t)^{\gamma_2-1} E_{\delta_2, \gamma_2}[-\lambda_n(-t)^{\delta_2}] \sin(\sqrt{\lambda_n}x) - \\ & - \sum_{n=1}^{\infty} \lambda_n \int_t^0 (z-t)^{\delta_2-1} E_{\delta_2, \delta_2}[-\lambda_n(z-t)^{\delta_2}] f_n(z) dz \sin(\sqrt{\lambda_n}x). \end{aligned}$$

To get rid of singularity in the integral we integrate by parts and considering Lemma 1.4.1 we have the following estimates  $t > 0$

$$\begin{aligned} |u_{xx}(x, t)| \leq & \sum_{n=1}^{\infty} \left( \frac{|\tau_{2n}| |t|^{\gamma_1-1} M}{1 + \lambda_n |t|^{\delta_1}} + \right. \\ & \left. + \frac{|f_{2n}(0)| |t|^{\delta_1}}{1 + \lambda_n |t|^{\delta_1}} + \int_0^t \frac{|t-s|^{\delta_1} M}{1 + \lambda_n |t-s|^{\delta_1}} |f'_{2n}(s)| ds \right), \end{aligned}$$

and as such for  $t < 0$

$$\begin{aligned} |u_{xx}(x, t)| \leq & \sum_{n=1}^{\infty} \left( \frac{|\lambda_n \varphi_n| - t|^{\gamma_2-2}}{1 + \lambda_n | - t|^{\delta_2}} + \frac{|\lambda_n \nu_n| - t|^{\gamma_2-1}}{1 + \lambda_n | - t|^{\delta_2}} \right) + \\ & + \sum_{n=1}^{\infty} \left( \frac{|f_{2n}(0)| | - t|^{\delta_2} M}{1 + \lambda_n | - t|^{\delta_2}} + \int_t^0 \frac{|z-t|^{\delta_2} M}{1 + \lambda_n |z-t|^{\delta_2}} |f'_{2n}(s)| ds \right), \end{aligned}$$

where  $\tau_n = \frac{\tau_{2n}}{\lambda_n}$ ,  $\varphi_n = \frac{\varphi_{2n}}{\lambda_n}$ ,  $\nu_n = \frac{\nu_{2n}}{\lambda_n}$ ,

$$f_{2n}(t) = \frac{2}{l} \begin{cases} \int_0^t f_{xx}(x, t) \sin(\sqrt{\lambda_n}x) dx, & t > 0, \\ \int_t^0 f_{xx}(x, t) \sin(\sqrt{\lambda_n}x) dx, & t < 0. \end{cases}$$

If  $f(x, t) \in C^{2,1}(0, l) \times (-T, T)$  and  $\tau(x), \varphi(x), \nu(x) \in C^2(0, l)$  and  $\tau'''(x), \varphi'''(x), \nu'''(x) \in L^2(0, l)$  which are required for  $t \rightarrow 0$ , then, the series representation of  $u_{xx}(x, t)$  is bounded with the convergent numerical series and from Weierstrass M-test the series of  $u_{xx}(x, t)$  converges uniformly in  $\Omega_1 \cup \Omega_2$ .

Finally, the uniform convergence of the series representation of  $D_{0\pm}^{(\alpha_i, \beta_i)\mu_i} u(x, t)$ ,  $i = \overline{1, 2}$  can be done similarly to the convergence of the series of  $u_{xx}(x, t)$  considering Eq.(3.3.1).

Moreover, according to (3.3.18)-(3.3.20) we can see that  $\tau(x)$ ,  $\varphi(x)$  and  $\nu(x)$  functions are written in terms of the given functions  $\psi(x)$  and  $f(x, t)$ . For that reason we write sufficient conditions for those given functions in order to show that all imposed conditions for  $\tau(x)$ ,  $\varphi(x)$  and  $\nu(x)$  are valid, i.e

$$\begin{aligned} \tau(x), \varphi(x), \nu(x) &\in C[0, l], \quad \tau(x), \varphi(x), \nu(x) \in C^2(0, l) \quad \text{and} \\ \tau'''(x), \varphi'''(x), \nu'''(x) &\in L^2(0, l), \quad f(x, t) \in C[0, l] \times [-T, T], \\ f(x, t) &\in C^{2,1}(0, l) \times (-T, T). \end{aligned}$$

If we find sufficient conditions for given functions in order to show the validity conditions of  $\nu(x)$ , it can be clearly seen that those sufficient conditions can be considered enough for showing that conditions for  $\tau(x)$ ,  $\varphi(x)$  are also valid automatically. Hence we have the following equality from (3.3.19)

$$\nu_n = \frac{-\lambda_n}{\Delta_n}(\psi_n + F_n) = -\frac{1}{\Delta_n \lambda_n \sqrt{\lambda_n}} \psi_{5n} - \frac{1}{\Delta_n \lambda_n \sqrt{\lambda_n}} F_{3n},$$

Since the given functions can be written in the form of a Fourier series and the last equality we have the following conditions for the given functions

$$\psi(x) \in C[0, l] \cap C^4(0, l) \quad \text{and} \quad \psi^{(5)}(x) \in L^2(0, l),$$

$$f(x, t) \in C[0, l] \times [-T, T] \cap C^{2,1}(0, l) \times (-T, T) \quad \text{and} \quad f_x^{(3)}(\cdot, t) \in L^2(0, l),$$

where we assume that  $\Delta_n \neq 0$ ,  $\psi(0) = \psi(l) = 0$ ,  $\psi''(0) = \psi''(l) = 0$ ,  $\psi^{(4)}(0) = 0$ ,  $\psi^{(4)}(l) = 0$ ,  $f(0, t) = f(l, t) = f_{xx}(0, t) = f_{xx}(l, t) = 0$  and we have used the following inequality

$$2 \left| \frac{1}{\Delta_n \sqrt{\lambda_n}} \psi_{5n} \right| \leq \frac{1}{\Delta_n^2 \lambda_n} + |\psi_{5n}|^2,$$

and Parseval's identity

$$\sum_{n=1}^{\infty} |\psi_{5n}|^2 = \|\psi^{(5)}\|^2,$$

$$\psi_n^{(5)} = \frac{2}{l} \int_0^l \psi^{(5)}(x) \sin(\sqrt{\lambda_n} x) dx,$$

$$F_{3n} = \int_0^T (T-z)^{\delta_1-1} E_{\delta_1, \delta_1}(-\lambda_n (T-z)^{\delta_1}) f_{3n}(z) dz -$$

$$- \int_{-T}^0 (z+T)^{\delta_2-1} E_{\delta_2, \delta_2}(-\lambda_n (z+T)^{\delta_2}) f_{3n}(z) dz,$$

$$|F_{3n}(t)| \leq |f_{3n}(0+)| \frac{MT^{\delta_1}}{1 + \lambda_n T^{\delta_1}} + \int_0^T |T-z|^{\delta_1} \frac{M}{1 + \lambda_n |T-z|^{\delta_1}} |f'_{3n}(z)| dz +$$

$$+ |f_{3n}(0-)| T^{\delta_2} \frac{M}{1 + \lambda_n T^{\delta_2}} + \int_{-T}^0 |z+T|^{\delta_2} \frac{M}{1 + \lambda_n |z+T|^{\delta_2}} |f'_{3n}(z)| dz,$$

where

$$f_{3n}(t) = \frac{2}{l} \int_0^l f_x^{(3)}(x, t) \cos(\sqrt{\lambda_n} x) dx.$$

$$f_{3n}(0) = \frac{2}{l} \int_0^l f_x^{(3)}(x, 0) \cos(\sqrt{\lambda_n} x) dx.$$

All in all, we have just proved the following theorem.

**Theorem 3.3.1.** *Assume that the following conditions hold:*

$\Delta_n \neq 0$ ;  $\psi(x) \in C[0, l] \cap C^4(0, l)$  such that  $\psi(0) = \psi(l) = 0$ ,  $\psi''(0) = \psi''(l) = 0$ ,  $\psi^{(4)}(0) = \psi^{(4)}(l) = 0$  and  $\psi^{(5)}(x) \in L^2(0, l)$ ;

$f(x, t) \in C[0, l] \times [-T, T] \cap C^{2,1}(0, l) \times (-T, T)$  such that  $f(0, t) = f(l, t) = 0$ ,  $f_{xx}(0, t) = f_{xx}(l, t) = 0$ ,  $f_x^{(3)}(\cdot, t) \in L^2(0, l)$ ;

*then, there exists the unique solution of the considered problem.*

## Conclusion of the Chapter 3

In section 3.1, the unique solvability of boundary value problem for mixed type partial differential equation is considered. As an interesting target for studying the boundary-value problems for mixed hyperbolic-parabolic type equations, it has been arising the necessity to generalize these kinds of problems. At the same time, while generalizing these problems by using fractional derivatives it is also important to use a more general definition of fractional differentiation. In this section, we investigated the boundary-value problem with consideration of both concerns aforementioned above.

We also generalized the results given in [5] by presenting a more general definition of regularized Caputo-like counterpart hyper-Bessel fractional differential operator at arbitrary starting point. Furthermore, the connection was established between the given data and the uniqueness and existence of the solution.

In section 3.2, we dealt with studying the nonlocal problem for the mixed type equation involving subdiffusion equation with regularized Caputo-like counterpart of hyper-Bessel differential operator and fractional wave equation with the bi-ordinal Hilfer's derivative. Through the properties Fourier-Bessel series and Mittag-Leffler function, the necessary conditions are found for the existence and uniqueness of the solution.

A distinctive side of this work than investigated in the Section 3.1 is that we consider the regularized Caputo-like counterpart of hyper-Bessel fractional differential operator which starting point is zero and the wave equation is generated the right-hand sided bi-ordinal Hilfer fractional derivative. Moreover, the nonlocal condition is used which expresses the equalization of summation of the values of Rieman-Liouville integrals of unknown function at discrete points of time to the value of this function at the final time.

The section 3.3 is presented as a combination of the sections 3.1 and 3.2. More precisely, the subdiffusion and fractional wave equations are generated by using the fractional bi-ordinal Hilfer derivative and non-homogeneous nonlocal condition is also used. We admit that by taking strict conditions for given data for showing the existence result we aimed to get away from singularities near zero. By the similar techniques applied in those sections the unique solvability of the nonlocal problem is investigated.



## Chapter 4

# Direct and inverse problems for the pseudo-parabolic equation with the bi-ordinal Hilfer fractional derivative

This chapter is devoted to study of some direct and inverse problems for the fractional pseudo-parabolic equations. Pseudo-parabolic equations are distinguished by the presence of a time derivative in the highest-order term and describe a variety of important physical processes. For example, the pseudo-parabolic equation is studied in the model which describes the energy of the isotropic material [21]. A concept regarding a non-simple material for which the conductive temperature and the thermodynamic temperature do not coincide. The non-stationary processes in semiconductors in the presence of sources can be analyzed by the pseudo-parabolic equation [117] and the filtration of the two-phase flow in porous media with the dynamic capillary pressure [11]. In [10] unique solvability of systems of time-fractional order pseudo-differential equations was concerned which can be modeled the various dynamical processes.

When it comes to the studying PDEs with degeneration, one can show the various methods which are applicable to use. For example, in [103] the initial boundary value problem is considered for the nonlinear degenerate parabolic

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equation with second order Volterra operator and the semi-discretization in time helps to use Rothe's method for finding approximate solution which has a application in the porous media equation and the integro-differential equation modeling memory effects.

Finding an effective and convenient methods for solving fractional partial differential equations (PDEs) is also an interesting part of the research among their applications. For example, in [116] it was noted the typical behaviour of the solution and explored which type of measurement is suited to recover, in a unique way, a space-dependent source  $f(x)$ . Most of the time the certain aspects of the equations and the properties of the fractional order derivative allow us to choose the methods to solve the problems. For example, the series method is often used to solve PDEs with any arbitrary order of derivatives and in this problem, it can be divided into two problems of solving ordinary differential equations.

Modeling the phenomena in physics or engineering often requires to study fractional order partial differential equation with variable coefficients. For example, in [6] authors considered fractional PDEs with the space-dependent coefficient and analyzed the uniqueness and existence of the solution with help of properties of the Legendre polynomials.

This chapter is based on the article [124], which was published in the journal *International Journal of Applied Mathematics*.

## 4.1 Unique solvability of the boundary value problem for the fractional Langevin-type partial differential equation

In physics, fractional order of Langevin equation plays an important role as a more detailed description of Brownian motion (See [95], sect. 15.5). When we consider the concept of the diffusion process which is associated with the random motion of particles in space, we see that normal diffusion and Brownian motion are related to the Langevin equation. The classical Langevin equation, in particular, by means of Newton's law treats the dynamics of a Brownian particle by combining the influence of Stokes fluid friction and temperature changes in the particle's proximity into a random force with appropriately assigned parameters.

Despite several other applications, it can be said that the Langevin equation itself is attractive too and many various differential equations have been considered in recent years. Langevin equation and the idea of further development and generalization of [6] were key motivation to investigate the present work. In addition to the physical application of this equation, we have focused on the unique solvability of the problem and the sufficient conditions for the existence of the solution.

In the current section, we are interested in investigating the following space-degenerate PDE

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} \left( D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t) - \frac{\partial}{\partial x} [(1 - x^2)u_x(x, t)] \right) = f(x, t) \quad (4.1.1)$$

in the domain  $\Omega = \{(x, t) : -1 < x < 1, 0 < t \leq T\}$ . Here  $D_{0+}^{(\alpha_s, \beta_s)\mu_s}$  is a left-sided bi-ordinal Hilfer fractional derivative defined in (1.3.4) when  $n = 1$  and  $0 < \alpha_s, \beta_s < 1, 0 \leq \mu_s \leq 1, s = \overline{1, 2}$ .

### 4.1.1 Statement of the problem and construction of a formal solution

**Problem A.** Find a solution  $u(x, t)$  of the equation (4.1.1) satisfying regularity conditions

$$t^{1-\gamma_2}u, \quad t^{1-\gamma_2}u_x \in C(\overline{\Omega}),$$

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u \in C(\Omega), \quad u_{xx} \in C(\Omega)$$

and initial condition

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u(x, t) = \psi(x), \quad -1 \leq x \leq 1, \quad (4.1.2)$$

and subject to the nonlocal condition

$$u(x, T) = \sum_{i=1}^m p_i I_{0+}^{q_i} D_{0+}^{\delta_2 + \gamma_1} u(x, \tau_i), \quad -1 < x < 1, \quad (4.1.3)$$

where  $\psi(x)$ ,  $f(x, t)$  are given functions and  $q_i > 0$ ,  $\delta_j = \beta_j + \mu_j(\alpha_j - \beta_j)$ ,  $\gamma_j = \beta_j + \mu_j(1 - \beta_j)$ ,  $j = \overline{1, 2}$ ,  $p_i \in \mathbb{R}$ ,  $0 < \tau_1 < \tau_2 < \dots < \tau_m \leq T$ , and also we assume that  $0 < \gamma_2 - \gamma_1 < \delta_2$ .

We investigate the solvability (uniqueness and existence) of this problem and present the solution in the form of Fourier-Legendre series as stated in the following theorem.

**Theorem 4.1.1.** *If  $\sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)} > 0$ ,  $\psi(x) \in C^1[-1, 1]$ ,  $\psi''(x) \in L^2(-1, 1)$ ,  $f(x, \cdot) \in C_{-1}^1[0, T]$  and  $f(\cdot, t) \in C[-1, 1]$ ,  $f_{xx}(\cdot, t) \in L^2(-1, 1)$ , then the Problem A has a unique solution which can be represented as*

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) P_k(x). \quad (4.1.4)$$

Here  $\lambda_k = k(k+1)$ ,  $k = 0, 1, 2, \dots$ ,  $\psi_k$  and  $f_k(t)$  are Fourier-Legendre coefficients of functions  $\psi(x)$  and  $f(x, t)$ , respectively,

$$u_k(t) = \psi_k t^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) + C_0 t^{\delta_2 + \gamma_1 - 1} E_{\delta_2, \delta_2 + \gamma_1}(-\lambda_k t^{\delta_2} +$$

$$+ \int_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2, \delta_2+\delta_1} [-\lambda_k(t-s)^{\delta_2}] f_k(s) ds,$$

$C_0$  is defined by the formula (4.1.14).

We note that the vector space  $C_{-1}$  is defined to be the set of all functions  $f(x)$ ,  $x > 0$ , expressible as  $f(x) = x^p f_1(x)$  for some real number  $p > -1$  and function  $f_1 \in C[0, \infty)$  and the vector space  $C_{-1}^1$  is defined to consist of all functions  $f(x)$ ,  $x > 0$ , such that  $f$  is one times differentiable and  $f' \in C_{-1}$  [72].

*Proof.* We intend to investigate this problem by applying the method of separation variables. From the equation (4.1.1) in the homogeneous case and considering  $u(x, t)$ ,  $u_x(x, t)$  are bounded at  $x = \pm 1$  which are come from regularity conditions, yield the following Legendre equation:

$$(1-x^2)X''(x) - 2xX'(x) + \lambda X(x) = 0 \quad (4.1.5)$$

and it has a bounded solution on  $[-1, 1]$  only if  $\lambda_k = k(k+1)$ ,  $k = 0, 1, 2, \dots$  and it is given by

$$X(x) = P_k(x) = \frac{1}{2^k \cdot k!} \frac{d^k (x^2 - 1)^k}{dx^k},$$

where  $P_k(x)$  is a Legendre polynomials.

It is known that (W. Kaplan [63], p. 511) the Legendre polynomials form a complete orthogonal system on  $[-1, 1]$  and any piece-wise continuous function  $g$  can be expressed in the form of Fourier-Legendre series with respect to the system  $\{P_k(x)\}$ :

$$g(x) = \sum_{k=0}^{\infty} c_k P_k(x), \quad c_k = \frac{(g, P_k)}{\|P_k\|^2} = \frac{2k+1}{2} \int_{-1}^1 g(x) P_k(x) dx.$$

Hence, we represent a sought function  $u(x, t)$  and the given function  $f(x, t)$  in the following forms:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) P_k(x), \quad (4.1.6)$$

$$f(x, t) = \sum_{k=0}^{\infty} f_k(t) P_k(x), \quad (4.1.7)$$

where  $u_k(t)$  are unknown and  $f_k(t)$  are the Fourier-Legendre coefficient of  $f(x, t)$  i.e,

$$f_k(t) = \frac{2k+1}{2} \int_{-1}^1 f(x, t) P_k(x) dx.$$

By substituting (4.1.6) and (4.1.7) into the equation (4.1.1) and initial condition (4.1.2) and nonlocal condition (4.1.3) one can obtain the following fractional differential equation

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} \left( D_{0+}^{(\alpha_2, \beta_2)\mu_2} + \lambda_k \right) u_k(t) = f_k(t), \quad (4.1.8)$$

with initial condition

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u_k(t) = \psi_k, \quad (4.1.9)$$

and nonlocal condition

$$u_k(T) = \sum_{i=1}^m p_i I_{0+}^{q_i} D_{0+}^{\delta_2 + \gamma_1} u_k(\tau_i), \quad (4.1.10)$$

where

$$\psi_k = \frac{2k+1}{2} \int_{-1}^1 \psi(x) P_k(x) dx.$$

Applying operator  $I_{0+}^{\delta_1}$  to both sides of (4.1.8) and using the Lemma 1.3.15, we obtain the following fractional differential equation

$$D_{0+}^{(\alpha_2, \beta_2)\mu_2} u_k(t) + \lambda_k u_k(t) = h(t), \quad (4.1.11)$$

where

$$h(t) = I_{0+}^{\delta_1} f_k(t) + \frac{C_0 t^{\gamma_1 - 1}}{\Gamma(\gamma_1)}.$$

The solution of the problem (4.1.11), (4.1.9) can be represented as presented in Lemma 1.3.16

$$u_k(t) = \psi_k t^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) + \int_0^t (t-\tau)^{\delta_2-1} E_{\delta_2, \delta_2}[-\lambda_k(t-\tau)^{\delta_2}] h(\tau) d\tau. \quad (4.1.12)$$

By substituting the expression of  $h(t)$  into the solution (4.1.12) and after some evaluations, we can rewrite the solution of (4.1.8) satisfying (4.1.9) as follows

$$u_k(t) = \psi_k t^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) + C_0 t^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1}(-\lambda_k t^{\delta_2}) + \int_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2, \delta_2+\delta_1}[-\lambda_k(t-s)^{\delta_2}] f_k(s) ds. \quad (4.1.13)$$

In order to find  $C_0$  we use the nonlocal condition (4.1.10), and we obtain

$$\begin{aligned} & \psi_k T^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k T^{\delta_2}) + C_0 T^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1}(-\lambda_k T^{\delta_2}) + \\ & + \int_0^T (T-s)^{\delta_2+\delta_1-1} E_{\delta_2, \delta_2+\delta_1}[-\lambda_k(T-s)^{\delta_2}] f_k(s) ds = \\ & = \psi_k \sum_{i=1}^m p_i \tau_i^{\gamma_2-\delta_2-\gamma_1+q_i-1} E_{\delta_2, \gamma_2-\gamma_1-\delta_2+q_i}(-\lambda_k \tau_i^{\delta_2}) - \\ & - C_0 \lambda_k \sum_{i=1}^m p_i \tau_i^{\delta_2+q_i-1} E_{\delta_2, \delta_2+q_i}(-\lambda_k \tau_i^{\delta_2}) + \\ & + f_k(0) \sum_{i=1}^m p_i \tau_i^{\delta_1-\gamma_1+q_i} E_{\delta_2, \delta_1-\gamma_1+q_i+1}(-\lambda_k \tau_i^{\delta_2}) + \\ & + \sum_{i=1}^m p_i \int_0^{\tau_i} (\tau_i-s)^{\delta_1-\gamma_1+q_i} E_{\delta_2, \delta_1-\gamma_1+q_i+1}[-\lambda_k(\tau_i-s)^{\delta_2}] f'_k(s) ds. \end{aligned}$$

From the last equality we find  $C_0$  as

$$C_0 = \frac{1}{\Delta_k} (B_k + F_k), \quad (4.1.14)$$

where

$$\begin{aligned}
 \Delta_k &= T^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1} \left( -\lambda_k T^{\delta_2} \right) + \lambda_k \sum_{i=1}^m p_i \tau_i^{\delta_2+q_i-1} E_{\delta_2, \delta_2+q_i} \left( -\lambda_k \tau_i^{\delta_2} \right), \\
 B_k &= \psi_k \left[ \sum_{i=1}^m p_i \tau_i^{\gamma_2-\delta_2-\gamma_1+q_i-1} E_{\delta_2, \gamma_2-\gamma_1-\delta_2+q_i} \left( -\lambda_k \tau_i^{\delta_2} \right) - \right. \\
 &\quad \left. - T^{\gamma_2-1} E_{\delta_2, \gamma_2} \left( -\lambda_k T^{\delta_2} \right) \right], \\
 F_k &= f_k(0) \sum_{i=1}^m p_i \tau_i^{\delta_1-\gamma_1+q_i} E_{\delta_2, \delta_1-\gamma_1+q_i+1} \left( -\lambda_k \tau_i^{\delta_2} \right) + \\
 &+ \sum_{i=1}^m p_i \int_0^{\tau_i} (\tau_i - s)^{\delta_1-\gamma_1+q_i} E_{\delta_2, \delta_1-\gamma_1+q_i+1} \left[ -\lambda_k (\tau_i - s)^{\delta_2} \right] f'_k(s) ds - \\
 &- \int_0^T (T - s)^{\delta_2+\delta_1-1} E_{\delta_2, \delta_2+\delta_1} \left[ -\lambda_k (T - s)^{\delta_2} \right] f_k(s) ds.
 \end{aligned}$$

We assume that  $\Delta_k \neq 0$  for any  $k$ , then (4.1.13) will be the solution of the problem (4.1.8), (4.1.9), (4.1.10).

Considering Lemma 1.4.5 we can show that

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \Delta_k &= \lim_{\lambda_k \rightarrow +\infty} \Delta_k = \lim_{|z_1| \rightarrow +\infty} T^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1}(z_1) - \\
 &- \lim_{|z_2| \rightarrow +\infty} \sum_{i=1}^m p_i \tau_i^{q_i-1} z_2 E_{\delta_2, \delta_2+q_i}(z_2) = \sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)}.
 \end{aligned}$$

Assuming  $\sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)} \neq 0$  then it confirms that  $\Delta_k \neq 0$  for any sufficiently large  $k$ .

According to Lemma 1.4.2 we can find lower bound of  $\Delta_k$  for any  $k$  as

$$\begin{aligned}
 \Delta_k &\geq \frac{T^{\delta_2+\gamma_1-1}}{\Gamma(\delta_2 + \gamma_1) + \Gamma(\gamma_1) \lambda_k T^{\delta_2}} + \sum_{i=1}^m \frac{p_i \lambda_k \tau_i^{\delta_2+q_i-1}}{\Gamma(\delta_2 + q_i) + \Gamma(q_i) \lambda_k \tau_i^{\delta_2}} \geq \\
 &\geq \sum_{i=1}^m \frac{p_i \lambda_k \tau_i^{\delta_2+q_i-1}}{\Gamma(\delta_2 + q_i) + \Gamma(q_i) \lambda_k \tau_i^{\delta_2}} = \sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)}.
 \end{aligned}$$



If  $\sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)} > 0$  for any  $k$ . Moreover, we may write upper bound of  $\frac{1}{\Delta_k}$  as

$$\frac{1}{\Delta_k} \leq M_1 = \frac{1}{\sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)}}.$$

Now we find upper bound of  $B_k$  and  $F_k$  by using well-known estimation of the Mittag-Leffler function given in Lemma 1.4.1:

$$|B_k| \leq |\psi_k| \left[ \sum_{i=1}^m |p_i| \frac{\tau_i^{\gamma_2 - \delta_2 - \gamma_1 + q_i - 1} M}{1 + \lambda_k \tau_i^{\delta_2}} + T^{\gamma_2 - 1} \frac{M}{1 + \lambda_k T^{\delta_2}} \right] \leq \frac{|\psi_k| M_2}{\lambda_k},$$

where  $M_2 = \sum_{i=1}^m |p_i| \tau_i^{\gamma_2 - 2\delta_2 - \gamma_1 - 1} + T^{\gamma_2 - \delta_2 - 1}$ ,

$$\begin{aligned} |F_k| &\leq |f_k(0)| \sum_{i=1}^m |p_i| \frac{\tau_i^{\delta_1 - \gamma_1 + q_i} M}{1 + \lambda_k \tau_i^{\delta_2}} + \\ &+ \sum_{i=1}^m |p_i| \int_0^{\tau_i} \frac{|\tau_i - s|^{\delta_1 - \gamma_1 + q_i} M}{1 + \lambda_k |\tau_i - s|^{\delta_2}} |f'_k(s)| ds + \\ &+ \int_0^T |T - s|^{\delta_2 + \delta_1 - 1} \frac{M}{1 + \lambda_k |T - s|^{\delta_2}} |f_k(s)| ds. \end{aligned}$$

If  $f'_k(s) \in C_{-1}[0, T]$ , then we may consider for  $k \rightarrow \infty$  that

$$|F_k| \leq M_3 < +\infty, \quad M_3 = \text{const} > 0.$$

Considering upper bounds of  $B_k$  and  $F_k$ , we have that

$$|C_0| \leq \frac{B_k + F_k}{\Delta_k} \leq \frac{|\psi_k| M_2 + M_3}{\lambda_k \sum_{i=1}^m \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)}} = \frac{M_4}{\lambda_k},$$

here we have assumed that  $\psi(x) \in C[-1, 1]$  and  $f(\cdot, t) \in C[-1, 1]$  and  $f(x, \cdot) \in C_{-1}^1[0, T]$ . Now we move on to proof of the uniqueness of the solution.

### 4.1.2 Uniqueness of the solution

Let there exist two solutions  $u_1(x, t)$  and  $u_2(x, t)$  of the main problem and consider the function  $u(x, t) = u_1(x, t) - u_2(x, t)$  which is a solution of the equation (4.1.1) in the homogeneous case with homogeneous initial conditions

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u(x, t) = 0, \quad -1 \leq x \leq 1. \quad (4.1.15)$$

Let us consider the following function

$$u_k(t) = \int_{-1}^1 u(x, t) P_k(x) dx, \quad k = 0, 1, 2, \dots, \quad (4.1.16)$$

Based on (4.1.16), we consider the function below

$$v_k(t) = \int_{\varepsilon-1}^{1-\varepsilon} u(x, t) P_k(x) dx, \quad k = 0, 1, 2, \dots, \quad (4.1.17)$$

where  $\varepsilon$  is very small positive number.

Applying the operator  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2}$  with respect to  $t$  to both sides of equality (4.1.17) and using the homogeneous equation corresponding with (4.1.1) yield that

$$\begin{aligned} D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} v_k(t) &= \int_{\varepsilon-1}^{1-\varepsilon} D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t) P_k(x) dx = \\ &= \int_{\varepsilon-1}^{1-\varepsilon} P_k(x) D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x(x, t)] dx, \end{aligned}$$

then integrating by parts twice the right side of the last equality and calculating the limit as  $\varepsilon \rightarrow 0$  give that

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} \left[ D_{0+}^{(\alpha_2, \beta_2)\mu_2} + \lambda_k \right] u(t) = 0.$$

Obviously, it can be shown that this equation with homogeneous condition (4.1.15) has only trivial solution  $u_k(t) \equiv 0$ ,  $t \in [0, T]$  and hence, from (4.1.16) we get

$$\int_{-1}^1 u(x, t) P_k(x) dx = 0, \quad k = 0, 1, 2, \dots.$$

Therefore, using the fact of completeness property of system  $\{P_k(x)\}$ , it is deduced that  $u(x, t) \equiv 0$  in  $\Omega$ , which proves the uniqueness of the considered problem.

### 4.1.3 Existence of the solution

In order to prove the existence of the solution in the form of (4.1.4), we need to show the uniform convergence of the series  $u(x, t)$ ,

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1 - x^2)u_x(x, t)] \quad \text{and} \quad D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t).$$

For  $k = 1, 2, 3, \dots$  the Legendre polynomials satisfy the following identities and relations [63]:

- 1)  $P'_{k+1}(x) - P'_{k-1}(x) = (2k+1)P_k(x)$ ,      2)  $\|P_k(\cdot)\|^2 = \frac{2}{2k+1}$ ,
- 3)  $P_k(1) = 1$ ,  $P_k(-1) = (-1)^k$ ,      4)  $|P_k(x)| \leq 1$ ,  $|x| \leq 1$ .

Let  $(f, g)$  be scalar product of the functions  $f$  and  $g$  in  $L^2(-1, 1)$ . Using above properties of the Legendre polynomials, we can write as

$$\begin{aligned} \psi_k &= \frac{2k+1}{2} \int_{-1}^1 \psi(x) P_k(x) dx = \\ &= \frac{2k+1}{2} \int_{-1}^1 \psi(x) \frac{1}{2k+1} [P'_{k+1}(x) - P'_{k-1}(x)] dx \end{aligned}$$

and integrating by parts

$$\psi_k = -\frac{1}{2} \int_{-1}^1 \psi(x) [P_{k+1}(x) - P_{k-1}(x)] dx = -\frac{1}{2} [(\psi', P_{k+1}) - (\psi', P_{k-1})].$$

Considering Schwartz inequality  $|(f, g)| \leq \|f\| \|g\|$ , we can write the estimation of  $\psi_k$ , where  $\|\cdot\| = \|\cdot\|_{L^2(-1,1)}$  :

$$\begin{aligned} |\psi_k| &\leq \frac{1}{2} |(\psi', P_{k+1})| + \frac{1}{2} |(\psi', P_{k-1})| \leq \\ &\leq \frac{1}{2} [\|\psi'\| \cdot \|P_{k+1}\| + \|\psi'\| \cdot \|P_{k-1}\|] \leq \\ &\leq \frac{1}{2} \|\psi'\| \left( \frac{\sqrt{2}}{(2k+3)^{\frac{1}{2}}} + \frac{\sqrt{2}}{(2k-1)^{\frac{1}{2}}} \right) \leq \frac{\|\psi'\| \sqrt{2}}{(2k-1)^{\frac{1}{2}}}. \end{aligned}$$

Repeating this process again we can obtain

$$|\psi_k| \leq \frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}} \|\psi''(\cdot)\|. \quad (4.1.18)$$

As a similar way, we write the estimation of  $f_k(t)$ :

$$|f_k(t)| \leq \frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}} \|f''_{xx}(\cdot, t)\|. \quad (4.1.19)$$

Considering above estimation for the Mittag-Leffler function we write the bound of  $u(x, t)$  by virtue of the properties of the Legendre polynomials

$$\begin{aligned} |u(x, t)| &\leq \sum_{k=0}^{\infty} \left[ |\psi_k| \frac{t^{\gamma_2-1} M}{1 + |\lambda_k| |t_2^\delta|} + |C_0| \frac{t^{\delta_2+\gamma_1-1} M}{1 + |\lambda_k| |t^{\delta_2}|} + \right. \\ &\quad \left. + \int_0^t |t-s|^{\delta_2+\delta-1} \frac{M}{1 + |\lambda_k| |t-s|^{\delta_2}} |f_k(s) ds| \right]. \end{aligned}$$

If  $\psi(x) \in C[-1, 1]$  and  $f(\cdot, t) \in C_{-1}[-1, 1]$ , we can show that the series of  $u(x, t)$  can be bounded by convergent series in  $\Omega$  domain and therefore by Weierstrass M-test the series representation of  $u(x, t)$  converges uniformly in  $\Omega$ .

After that by using properties of the Legendre polynomials we show uniform convergence of the series of  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x]$  which is represented as follows:

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x] = - \sum_{k=0}^{\infty} \lambda_k D_{0+}^{(\alpha_1, \beta_1)\mu_1} u_k(t) P_k(x) =$$

$$\begin{aligned}
 &= - \sum_{k=0}^{\infty} \lambda_k \left[ \psi_k t^{\gamma_2 - \delta_1 - 1} E_{\delta_2, \gamma_2 - \delta_1}(-\lambda_k t^{\delta_2}) + C_0 t^{\delta_2 + \gamma_1 - \delta_1 - 1} E_{\delta_2, \delta_2 + \gamma_1 - \delta_1}(-\lambda_k t^{\delta_2}) + \right. \\
 &\quad \left. + \int_0^t (t-s)^{\delta_2 - 1} E_{\delta_2, \delta_2}[-\lambda_k (t-s)^{\delta_2}] f_k(s) ds \right] P_k(x).
 \end{aligned}$$

By means of properties of the Legendre polynomials and the upper bounds of the Mittag-Leffler function presented by the Lemma 1.4.1 and above estimations for given functions, we obtain the following estimate

$$\begin{aligned}
 &\left| D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x] \right| \leq \sum_{k=0}^{\infty} |D_{0+}^{(\alpha_1, \beta_1)\mu_1} u_k(t) \lambda_k P_k(x)| \leq \\
 &\leq \sum_{k=0}^{\infty} \lambda_k \left[ |\psi_k| \frac{t^{\gamma_2 - \delta_1 - 1} M}{1 + \lambda_k |t^{\delta_2}|} + |C_0| \frac{t^{\delta_2 + \gamma_1 - \delta_1 - 1} M}{1 + \lambda_k |t^{\delta_2}|} \right] + \\
 &+ \sum_{k=0}^{\infty} \lambda_k \int_0^t |t-z|^{\delta_2 - 1} \frac{M}{1 + \lambda_k ||t-z|^{\delta_2}|} |f_k(z)| dz \leq \\
 &\leq \sum_{k=0}^{\infty} \left[ \frac{MT^{\gamma_2 - \delta_1 - \delta_2 - 1} 4\sqrt{2}}{(2k-3)^{\frac{3}{2}}} \|\psi''(x)\| + \frac{MM_4 T^{\gamma_1 - \delta_1 - 1}}{\lambda_k} \right] + \\
 &+ \sum_{k=0}^{\infty} \int_0^t |t-z|^{\delta_2 - 1} \frac{M\lambda_k}{1 + \lambda_k ||t-z|^{\delta_2}|} \frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}} \|f''_{xx}(\cdot, z)\| dz,
 \end{aligned}$$

where  $\lambda_k = k(k+1)$ .

From the last inequalities one can show that the series of representation of  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x]$  is bounded by a convergent series which implies that it is convergent uniformly according to the Weierstrass M-test in  $\Omega$ .

Finally,  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t)$  which can be represented by the equation

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t) = D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x(x, t)] + f(x, t)$$

and its uniform convergence can be shown as a similar way which we have done before to the  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x]$ .

All in all, we have proved the Theorem 4.1.1. □

## 4.2 Inverse source problem of determining a time-dependent source in a fractional Langevin type PDE

On the one hand, studying inverse source problems have been an important target as research for mathematicians due to the applications in science and engineering [108], [15].

On the other hand, showing a solution's global (in time) existence and uniqueness is a complicated task. Another interesting thing is that the goal in solving inverse source problems is their solvability and description of a constructive algorithm for finding a solution. We suggest the references [45], [101], [105] for readers to get a piece full of detailed information about some methods of solving inverse source problems devoted to determining  $t$ -dependent factor in the source such as analytical and numerical techniques.

In this section we are also concerning with studying the time dependent inverse source problem for the equation (4.1.1) and we will take another more favorable condition in order to facilitate calculations instead of non-local condition (4.1.3).

Letting the source term have the form  $f(x, t) = a(t)h(x)$ , then the inverse problem consists of determining a source term  $a(t)$  and  $u(x, t)$  (the temperature distribution in the heat process), from the initial data  $\psi(x)$  (initial temperature in heat process) and boundary conditions which come from regularity conditions.

We refer the references that some researches about determining time dependent coefficient in the source term of PDEs by using different methods [106]

### 4.2.1 Mathematical setting

Let  $\gamma_1 < \gamma_2 - \delta_2$ ,  $T > 0$  arbitrary fixed time and  $\Omega = \{(x, t) : -1 < x < 1, 0 < t \leq T\}$ .

The inverse source problem (ISP) here is to find a pair  $\{u(x, t), a(t)\}$  functions for given  $h(x)$ ,  $\psi(x)$ ,  $\varphi(x)$  such that

$$\begin{aligned} D_{0+}^{(\alpha_1, \beta_1)\mu_1} \left( D_{0+}^{(\alpha_2, \beta_2)\mu_2} u(x, t) - \frac{\partial}{\partial x} [(1 - x^2)u_x(x, t)] \right) &= h(x)a(t), \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u(x, t) &= \psi(x), \quad -1 \leq x \leq 1, \\ \lim_{t \rightarrow 0+} I_{0+}^{1-\gamma_1-\delta_2} u(x, t) &= \varphi(x), \quad -1 \leq x \leq 1, \end{aligned} \quad (4.2.1)$$

We provide the over-determination condition as a way to make the inverse problem uniquely solved:

$$\int_{-1}^1 u(x, t) dx = E(t), \quad (4.2.2)$$

where  $E(t) \in AC^2([0, T], \mathbb{R})$ .

We also consider the following regularity conditions for the solution of inverse source problem (4.2.1), (4.2.2)

$$t^{1-\delta_2-\gamma_1} u \in C(\overline{\Omega}), \quad t^{1-\delta_2-\gamma_1} u_x \in C(\overline{\Omega}), \quad t^{1-\delta_2-\gamma_1} a(t) \in C[0, T],$$

$$D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u \in C(\Omega), \quad u_{xx} \in C(\Omega).$$

**Theorem 4.2.1.** *Let  $\gamma_1 < \gamma_2 - \delta_2$ ,  $0 < \left( \int_{-1}^1 h(x) dx \right)^{-1} < M$  such that  $h^{(4)}(x) \in L^2(-1, 1)$ ,  $\psi''(x) \in L^2(-1, 1)$ ,  $\varphi''(x) \in L^2(-1, 1)$ ,  $E''(t) \in L^1(0, T)$ , then the unique solution of the inverse source problem (4.2.1)-(4.2.2) exists.*

*Proof.* **Existence of a solution to the ISP**

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (4.2.1), (4.2.2) for arbitrary  $a(t) \in$

$C[0, T]$

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) P_k(x), \quad (4.2.3)$$

where

$$\begin{aligned} u_k(t) = & \psi_k t^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) + \varphi_k t^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1}(-\lambda_k t^{\delta_2} + \\ & + h_k \int_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2, \delta_2+\delta_1}[-\lambda_k(t-s)^{\delta_2}] a(s) ds, \end{aligned} \quad (4.2.4)$$

$$\text{with } \psi_k = \frac{2k+1}{2} \int_{-1}^1 \psi(x) P_k(x) dx, \quad \varphi_k = \frac{2k+1}{2} \int_{-1}^1 \varphi(x) P_k(x) dx,$$

$$h_k = \frac{2k+1}{2} \int_{-1}^1 h(x) P_k(x) dx, \quad k = 0, 1, 2, \dots$$

Taking fractional derivative  $D_{0+}^{(\alpha_2, \beta_2)\mu_2}$  and  $D_{0+}^{(\alpha_1, \beta_1)\mu_1}$  under the integral sign of the over-determination condition (4.2.2) and in view of the equation in (4.2.1), we obtain

$$a(t) = \left( \int_{-1}^1 h(x) dx \right)^{-1} \cdot D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} E(t). \quad (4.2.5)$$

Substituting (4.2.5) into (4.2.4) and assuming that  $D_{0+}^{(\alpha_2, \beta_2)\mu_2} E(t) \in L^1(0, T)$ ,  $E(t) \in C[0, T]$  and  $E'(t) \in L^1(0, T)$  step by step, we get

$$\begin{aligned} u_k(t) = & \psi_k t^{\gamma_2-1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) + \varphi_k t^{\delta_2+\gamma_1-1} E_{\delta_2, \delta_2+\gamma_1}(-\lambda_k t^{\delta_2} + \\ & + h_k h^* \left( E(t) - \lambda_k \int_0^t (t-s)^{\delta_2-1} E_{\delta_2, \delta_2}[-\lambda_k(t-s)^{\delta_2}] E(s) ds \right), \end{aligned} \quad (4.2.6)$$

$$\text{where } h^* = \left( \int_{-1}^1 h(x) dx \right)^{-1}.$$



Now let us establish necessary condition for  $E(t)$  in order to show the existence of (4.2.5) and here we might take stronger condition for  $E(t)$ .

Considering Definition 1.3.11 we make sure that  $I_{0+}^{1-\gamma_1} \left( D_{0+}^{(\alpha_2, \beta_2)\mu_2} E(t) \right) \in AC[0, T]$  or  $D_{0+}^{(\alpha_2, \beta_2)\mu_2} E(t) \in AC[0, T]$ . From the last condition we derive that  $E'(t) \in C[0, T]$  and  $E''(t) \in L^1(0, T)$ . These conditions ensure that  $a(t) \in C[0, T]$ .

From the properties of Legendre polynomials discussed in the section 4.1, we recall that

$$|h_k| \leq \frac{4\sqrt{2}}{(2k-3)^{3/2}} \|h''(\cdot)\|, \quad |h_k| \leq \frac{6\sqrt{2}}{(2k-7)^{7/2}} \|h^{(4)}(\cdot)\|.$$

where  $\|\cdot\|$  is a norm of  $L^2(-1, 1)$ .

Now, by taking estimate for the series  $t^{1-\delta_2-\gamma_1}u(x, t)$ , for all  $t \in [0, T]$  we get

$$\begin{aligned} \left| t^{1-\delta_2-\gamma_1} u(x, t) \right| &\leq \sum_{k=0}^{\infty} |u_k(t)|^2 |P_k(x)| \leq \\ &\sum_{k=0}^{\infty} |\psi_k| \left| t^{\gamma_2-\delta_2-\gamma_1} E_{\delta_2, \gamma_2}(-\lambda_k t^{\delta_2}) \right|^2 + \sum_{k=0}^{\infty} |\varphi_k| \left| E_{\delta_2, \delta_2-\gamma_2+\gamma_1+1}(-\lambda_k t^{\delta_2}) \right| + \\ &\sum_{k=0}^{\infty} |h_k| |h^*| \left( |E(t)|^2 + |E(0)|^2 \lambda_k \left| t^{\delta_2} E_{\delta_2, \delta_2+1}(-\lambda_k t^{\delta_2}) \right|^2 \right) + \\ &+ \sum_{k=0}^{\infty} |h_k| |h^*| \lambda_k \int_0^t |(t-s)^{\delta_2} E_{\delta_2, \delta_2+1}[-\lambda_k(t-s)^{\delta_2}]| |E'(s)| ds \leq \\ &\leq \sum_{k=0}^{\infty} |\psi_k| \left( \frac{M t^{\gamma_2-1}}{1 + \lambda_k t^{\delta_2}} \right) + \sum_{k=0}^{\infty} |\varphi_k| \left( \frac{M t^{\delta_2+\gamma_1-1}}{1 + \lambda_k t^{\delta_2}} \right) + \\ &+ \sum_{k=0}^{\infty} |h_k| |h^*| \left( |E(t)| + |E(0)| \lambda_k \left( \frac{M t^{\delta_2}}{1 + \lambda_k t^{\delta_2}} \right) \right) + \\ &+ \sum_{k=0}^{\infty} |h_k| |h^*| \int_0^t \left( \frac{M \lambda_k (t-s)^{\delta_2}}{1 + \lambda_k (t-s)^{\delta_2}} \right) |E'(s)| ds \leq \end{aligned}$$

$$\begin{aligned}
 &\leq \|\psi''(\cdot)\| \sum_{k=0}^{\infty} \frac{4\sqrt{2}}{(2k-3)^{3/2}} \left( \frac{Mt^{\gamma_2-1}}{1+\lambda_k t^{\delta_2}} \right) + \|\varphi''(\cdot)\| \sum_{k=0}^{\infty} \frac{4\sqrt{2}}{(2k-3)^{3/2}} \left( \frac{Mt^{\delta_2+\gamma_1-1}}{1+\lambda_k t^{\delta_2}} \right) + \\
 &\quad + \|h''(\cdot)\| \sum_{k=0}^{\infty} \frac{|h^*|4\sqrt{2}}{(2k-3)^{3/2}} \left( |E(t)| + |E(0)| \left( \frac{M\lambda_k t^{\delta_2}}{1+\lambda_k t^{\delta_2}} \right) \right) + \\
 &\quad + \|h''(\cdot)\| \sum_{k=0}^{\infty} \frac{|h^*|4\sqrt{2}}{(2k-3)^{3/2}} \int_0^t \left( \frac{M\lambda_k(t-s)^{\delta_2}}{1+\lambda_k(t-s)^{\delta_2}} \right) |E'(s)| ds.
 \end{aligned}$$

We presume  $E'(t) \in C[0, T]$  and considering Weierstrass M- test one can see that the series representation of  $t^{1-\delta_2-\gamma_1}u(x, t)$  converges uniformly.

Now, for the second term of the equation (4.1.1) we have

$$\begin{aligned}
 D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x] &= - \sum_{k=0}^{\infty} \lambda_k D_{0+}^{(\alpha_1, \beta_1)\mu_1} u_k(t) P_k(x) = \\
 &= - \sum_{k=0}^{\infty} \lambda_k \left[ \psi_k t^{\gamma_2-\delta_1-1} E_{\delta_2, \gamma_2-\delta_1}(-\lambda_k t^{\delta_2}) + \varphi_k t^{\delta_2+\gamma_1-\delta_1-1} E_{\delta_2, \delta_2+\gamma_1-\delta_1}(-\lambda_k t^{\delta_2}) + \right. \\
 &\quad + h_k h^* D_{0+}^{(\alpha_1, \beta_1)\mu_1} E(t) - \lambda_k h_k h^* E(0) t^{\delta_2-\delta_1} E_{\delta_2, \delta_2-\delta_1+1}(-\lambda_k t^{\delta_2}) \\
 &\quad \left. - \lambda_k h_k h^* \int_0^t (t-s)^{\delta_2-\delta_1} E_{\delta_2, \delta_2-\delta_1+1}[-\lambda_k(t-s)^{\delta_2}] E'(s) ds \right] P_k(x).
 \end{aligned}$$

By taking estimate with help of properties of Legendre polynomials and Mittag-Leffler function, we get

$$\begin{aligned}
 \left| D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x(\cdot, t)] \right| &\leq \\
 &\sum_{k=0}^{\infty} \left[ |\lambda_k \psi_k| \left( \frac{Mt^{\gamma_2-\delta_1-1}}{1+\lambda_k t^{\delta_2}} \right) + |\lambda_k \varphi_k| \left( \frac{Mt^{\delta_2+\gamma_1-\delta_1-1}}{1+\lambda_k t^{\delta_2}} \right) + \right. \\
 &\quad \left. + |\lambda_k h_k| |h^* D_{0+}^{(\alpha_1, \beta_1)\mu_1} E(t)| + |\lambda_k h_k h^*| |\lambda_k E(0)| \left( \frac{Mt^{\delta_2-\delta_1}}{1+\lambda_k t^{\delta_2}} \right) + \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\lambda_k h_k h^*| |\lambda_k E'(0)| \left( \frac{M t^{\delta_2+1-\delta_1}}{1 + \lambda_k t^{\delta_2}} \right) + \\
 & \quad + |\lambda_k^2 h_k h^*| \int_0^t \left( \frac{M(t-s)^{\delta_2+1-\delta_1}}{1 + \lambda_k(t-s)^{\delta_2}} \right) |E''(s)| ds \Big] \leq \\
 & \leq \sum_{k=0}^{\infty} \frac{\|\psi''(\cdot)\| 4\sqrt{2}}{(2k-3)^{3/2}} \left( \frac{\lambda_k M t^{\gamma_2-\delta_1-1}}{1 + \lambda_k t^{\delta_2}} \right) + \sum_{k=0}^{\infty} \frac{\|\varphi''(\cdot)\| 4\sqrt{2}}{(2k-3)^{3/2}} \left( \frac{\lambda_k M t^{\gamma_2-\delta_1-1}}{1 + \lambda_k t^{\delta_2}} \right) + \\
 & \quad \sum_{k=0}^{\infty} \frac{6\sqrt{2}\lambda_k \|h^{(4)}(\cdot)\|}{(2k-7)^{7/2}} \left( |h^* D_{0+}^{(\alpha_1, \beta_1)\mu_1} E(t)| + |E(0)| \left( \frac{M \lambda_k t^{\delta_2-\delta_1}}{1 + \lambda_k t^{\delta_2}} \right) \right) + \\
 & \quad \sum_{k=0}^{\infty} \frac{6\sqrt{2}\lambda_k \|h^{(4)}(\cdot)\|}{(2k-7)^{7/2}} \left( |E'(0)| \left( \frac{M \lambda_k t^{\delta_2+1-\delta_1}}{1 + \lambda_k t^{\delta_2}} \right) + \right. \\
 & \quad \left. + |h^*| \int_0^t \left( \frac{M \lambda_k (t-s)^{\delta_2+1-\delta_1}}{1 + \lambda_k(t-s)^{\delta_2}} \right) |E''(s)| ds \right)
 \end{aligned}$$

If  $\psi''(x) \in L^2(-1, 1)$ ,  $\varphi''(x) \in L^2(-1, 1)$ ,  $h^{(4)}(x) \in L^2(-1, 1)$  and  $E''(t) \in L^1(0, T)$ , then the series representation of  $\|D_{0+}^{(\alpha_1, \beta_1)\mu_1} \frac{\partial}{\partial x} [(1-x^2)u_x]\|$  converges uniformly in  $\Omega$ . Finally, in a similar way, one can show that the series  $D_{0+}^{(\alpha_1, \beta_1)\mu_1} D_{0+}^{(\alpha_2, \beta_2)\mu_2} u$  is uniformly convergent in  $\Omega$ .

This proves existence of solution of ISP.

The uniqueness of the ISP can be shown by using the fact of completeness of the system of Legendre polynomials  $P_k(x)$  in  $L^2(-1, 1)$ .  $\square$

## 4.3 Direct and inverse source problem for 2D Landau Hamiltonian operator

### 4.3.1 Formulation of a problem

In this section, we consider a non-relativistic particle with mass  $m$  and electric charge  $e$  moving in a given electromagnetic field and concentrate on the 2D

version. Usually vector potential  $\mathbb{A}$  is used for describing the electromagnetic field in the plane. One of the classical models in quantum physics is the Landau Hamiltonian, which describes the behavior of a quantum particle in two dimensions under the influence of a constant magnetic field and it was introduced in late 1920-s (see [69]). On the Euclidean  $xy$ -plane interacting with a perpendicular homogeneous electromagnetic field, the following Hamiltonian operator determines the dynamics of a particle with mass  $m$  and charge  $e$ :

$$\mathcal{H}_0 := \frac{1}{2m} \left( ih\nabla - \frac{e}{c}\mathbb{A} \right)^2, \quad (4.3.1)$$

where  $i$  denotes the imaginary unit,  $h$  is Planck's constant,  $c$  is the speed of light. We denote by  $2B > 0$  the strength of the magnetic field and the symmetric gauge can be selected by

$$\mathbb{A} := \frac{\mathbf{r}}{2} \times \mathbb{B} = (-By, Bx),$$

where  $\mathbf{r} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2$ .

In  $\Omega = \{(t, x) : t \in (0, T), x \in \mathbb{R}^2\}$ , we consider the following Cauchy-type problem for the fractional pseudo-parabolic equation involving the bi-ordinal Hilfer's derivative and a Landau Hamiltonian operator with initial-type conditions

$$\begin{cases} D_t^{(\alpha, \beta)\mu} u(t, x) + (aD_t^{(\alpha, \beta)\mu} + q)\mathcal{H}u(t, x) = f(t, x), & (t, x) \in \Omega, \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} u(t, x) = u_0(x), & x \in \mathbb{R}^2. \end{cases} \quad (4.3.2)$$

where  $a, q \in \mathbb{R}^+$ ,  $0 < \alpha, \beta \leq 1$ ,  $0 \leq \mu \leq 1$  and  $D_t^{(\alpha, \beta)\mu}$  is the bi-ordinal Hilfer's fractional derivative of orders  $\alpha$  and  $\beta$  of type  $\mu$  as stated in (1.3.3),  $\mathcal{H}$  is a Landau Hamiltonian operator acting on the  $L^2(\mathbb{R}^2)$  defined by

$$\mathcal{H} = \frac{1}{2} \left( \left( i \frac{\partial}{\partial x} - By \right)^2 + \left( i \frac{\partial}{\partial y} - Bx \right)^2 \right), \quad (4.3.3)$$

which can be obtained from (4.3.1) by setting  $m = h = c = e = 1$ .

The spectrum of  $\mathcal{H}$  contains the infinite number of eigenvalues with the infinite multiplicity of the form (see [33], [70])

$$\lambda_n = (2n + 1)B, n = 0, 1, 2, \dots,$$

and those eigenvalues are called the Euclidean Landau levels.

Let us denote the eigenspace of  $\mathcal{H}$  corresponding to the eigenvalue  $\lambda_n$  by  $\mathcal{A}_n(\mathbb{R}^2)$ , i.e.

$$\mathcal{A}_n(\mathbb{R}^2) = \{\phi \in L^2(\mathbb{R}^2), \mathcal{H}\phi = \lambda_n\phi\}.$$

The eigenfunctions corresponding  $\lambda_n$  can be designated by

$$e_\xi^k := e_{j,n}^k \quad \text{for } \xi = (j, n), \quad j, n = 0, 1, 2, \dots; k = 1, 2. \quad (4.3.4)$$

In the references [1], [44], the following functions are presented as the orthonormal basis for  $\mathcal{A}_n(\mathbb{R}^2)$ :

$$\left\{ \begin{aligned} e_{k,n}^1(x, y) &= c_{k,n} \sqrt{\frac{n!}{(n-k)!}} B^{\frac{k+1}{2}} \exp\left(-\frac{B(x^2+y^2)}{2}\right) \times \\ &\quad (x + iy)^k L_n^{(k)}(B(x^2 + y^2)), 0 \leq k, \\ e_{j,n}^2(x, t) &= c_{jn} \sqrt{\frac{j!}{(n+n)!}} B^{\frac{n-1}{2}} \exp\left(-\frac{B(x^2+y^2)}{2}\right) \times \\ &\quad (x - iy)^n L_j^{(n)}(B(x^2 + y^2)), 0 \leq j, \end{aligned} \right. \quad (4.3.5)$$

where  $c_{kn}, c_{jn}$  are the normalizing coefficients,  $L_n^{(\alpha)}$  is the Laguerre polynomial defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n (-1)^k C_{n+\alpha}^{n-k} \frac{x^k}{k!}, \quad \alpha > -1.$$

It should be noted that while the physical aspects of the Landau Hamiltonian operator are being studied, the importance of studying the mathematical equations which are involved with this operator is increasing. As an example, let us consider the reference [97] which devoted studying the following Cauchy

problem:

$$\begin{cases} \partial_t^2 u(t, x) + a(t)[\mathcal{H} + q(t)]u(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^2, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2, \\ \partial_t u(0, t) = u_1(x), & x \in \mathbb{R}^2, \end{cases} \quad (4.3.6)$$

and the theorem about the existence, uniqueness and consistency of the solution for the Landau Hamiltonian on  $\mathbb{R}^2$  is proved. The description of the notion of very weak solution for the wave equation for the Landau Hamiltonian is also given. Moreover, it is established the well-posedness results for many types of the wave equation for the Landau Hamiltonian similar to (4.3.6) with irregular electromagnetic field and similarly irregular velocity. Detailed information about the investigations of the Fourier analysis for the Landau Hamiltonian can be also found in these references [98], [99], [100].

### 4.3.2 Fourier analysis for the Landau Hamiltonian

In this subsection we recall the auxiliary results of the global Fourier analysis for the Landau Hamiltonian that has been developed in [99], [100].

Let  $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) := \text{Dom}(\mathcal{H}^{\infty})$  signify the space of test functions for  $\mathcal{H}$  which has the form below

$$\text{Dom}(\mathcal{H}^{\infty}) := \bigcap_{k=1}^{\infty} \text{Dom}(\mathcal{H}^k)$$

where  $\text{Dom}(\mathcal{H}^k)$  is the domain of the operator  $\mathcal{H}^k$  identified by

$$\text{Dom}(\mathcal{H}^k) := \{f \in L^2(\mathbb{R}^2) : \mathcal{H}^j f \in \text{Dom}(\mathcal{H}), j = 0, 1, 2, \dots, k-1\}.$$

The Frechet topology of  $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$  is given by the family of norms

$$\|\varphi\|_{C_{\mathcal{H}}^k} := \max_{j \leq k} \|\mathcal{H}^j \varphi\|_{L^2(\mathbb{R}^2)}, \quad k \in \mathbb{N}_0, \quad \varphi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2). \quad (4.3.7)$$

We also recall the definition of the space of linear continuous functionals on  $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$  by

$$\mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) := \mathcal{L}(C_{\mathcal{H}}^{\infty}(\mathbb{R}^2), \mathbb{C})$$

which is called the space of  $\mathcal{H}$ -distributions and here the continuity is considered in terms of topology (4.3.7). It is concluded that  $\omega(\varphi) = \langle \omega, \varphi \rangle$  is written for  $\omega \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$  and  $\varphi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ .

The functional which is an  $\mathcal{H}$ -distributions enucleated as

$$\varphi \mapsto \int_{\mathbb{R}^2} \psi(x) \varphi(x) dx \quad \varphi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$$

for any  $\psi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ , which gives an embedding  $\psi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) \hookrightarrow \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$ . It can be clearly seen that using the notation (4.3.4) opens a way to group the pairs of eigenfunctions of the Landau Hamiltonian (4.3.5). After this notion, it is expedient to recall the following definitions presented in [100] step by step.

Let  $\mathcal{S}(\mathbb{N}^2)$  denote the space of rapidly decaying functions  $\varphi : \mathbb{N}^2 \rightarrow \mathbb{C}^{2 \times 2}$  of the form

$$\varphi := \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}.$$

That is,  $\varphi \in \mathcal{S}(\mathbb{N}^2)$  if for any  $M < \infty$  there exists a constant  $C_{\varphi, M}$  such that

$$|\varphi(\xi)| \leq C_{\varphi, M} \langle \xi \rangle^{-M}$$

holds for all  $\xi \in \mathbb{N}^2$ , where we denote

$$\langle \xi \rangle := |\sqrt{\lambda_{\xi_2}}| = \left| \sqrt{(2\xi_2 + 1)B} \right|.$$

The topology on  $\mathcal{S}(\mathbb{N}^2)$  is given by the semi-norms  $p_k$ , where  $k \in \mathbb{N}_0$  and

$$p_k(\varphi) := \sup_{\xi \in \mathbb{N}^2} \langle \xi \rangle^k |\varphi(\xi)|.$$

We now define the  $\mathcal{H}$ -Fourier transform on  $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$  as the mapping

$$(\mathcal{F}_{\mathcal{H}} f)(\xi) = (f \mapsto \hat{f}) : C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{N}^2)$$

by the formula

$$\hat{f}(\xi) := (\mathcal{F}_{\mathcal{H}} f)(\xi) = \int_{\mathbb{R}^2} f(x) \overline{e_{\xi}(x)} dx \quad (4.3.8)$$

where

$$e_\xi(x) = \begin{pmatrix} e_\xi^1(x) & 0 \\ 0 & e_\xi^2(x) \end{pmatrix}.$$

The Fourier transform  $\mathcal{F}_\mathcal{H}$  is a bijective homeomorphism from  $C_\mathcal{H}^\infty(\mathbb{R}^2)$  to  $\mathcal{S}(\mathbb{N}^2)$ . Its inverse

$$\mathcal{F}_\mathcal{H}^{-1} : \mathcal{S}(\mathbb{N}^2) \rightarrow C_\mathcal{H}^\infty(\mathbb{R}^2)$$

is given by

$$(\mathcal{F}_\mathcal{H}^{-1}h)(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr}(h(\xi)e_\xi(x)), \quad h \in \mathcal{S}(\mathbb{N}^2), \quad (4.3.9)$$

so that the Fourier inversion formula becomes

$$f(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr}(\hat{f}(\xi)e_\xi(x)) \quad \text{for all } f \in C_\mathcal{H}^\infty(\mathbb{R}^2). \quad (4.3.10)$$

The Plancherel's identity taking the form

$$\|f\|_{L^2(\mathbb{R}^2)} = \left( \sum_{\xi \in \mathbb{N}^2} \|\hat{f}(\xi)\|_{HS}^2 \right)^{1/2} =: \|\hat{f}\|_{\ell^2(\mathbb{N}^2)}, \quad (4.3.11)$$

which we take as the definition of the norm on the Hilbert space  $\ell^2(\mathbb{N}^2)$ , and where  $\|\hat{f}(\xi)\|_{HS}^2 = \text{Tr}(\hat{f}(\xi)\overline{\hat{f}(\xi)})$  is the Hilbert-Schmidt norm of the matrix  $\hat{f}(\xi)$ .

Since the test functions and distributions on  $\mathbb{R}$  characterised with the help of their Fourier coefficients, we have

$$f \in C_\mathcal{H}^\infty(\mathbb{R}^2) \Leftrightarrow \forall N \exists C_N \text{ such that } \|\hat{f}(\xi)\|_{HS}^2 \leq C_N \langle \xi \rangle^{-N} \text{ for all } \xi \in \mathbb{N}^2$$

Also for distributions, we have

$$u \in \mathcal{D}'_\mathcal{H}(\mathbb{R}^2) \Leftrightarrow \exists M \exists C \text{ such that } \|\hat{u}(\xi)\|_{HS} \leq C \langle \xi \rangle^M \text{ for all } \xi \in \mathbb{N}^2.$$

The operator  $\mathcal{H}$  is formally self-adjoint, therefore its symbol  $\sigma_\mathcal{H}(\xi)$  is independent of  $x$  and can be diagonalized by a choice of the basis in the  $\mathbb{R}^2$ .



Moreover, the operator  $\mathcal{H}$  is positive. Therefore, without loss of generality, we can always write

$$\sigma_{\mathcal{H}}(\xi) = \begin{pmatrix} \nu_1^2(\xi) & 0 \\ 0 & \nu_2^2(\xi) \end{pmatrix}, \quad (4.3.12)$$

for some  $\nu_j(\xi) \geq 0$ . Indeed, we have  $\nu_j^2(\xi) = B(1 + 2\xi_2)$  for  $j = 1, 2$ .

Consequently we define Sobolev space  $H_{\mathcal{H}}(G)$  associated to  $\mathcal{H}$ . Thus, for any  $s \in \mathbb{R}$ , we set

$$H_{\mathcal{H}}^s(G) := \left\{ f \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) : \mathcal{H}^{s/2} f \in L^2(\mathbb{R}^2) \right\}, \quad (4.3.13)$$

with the norm  $\|f\|_{H_{\mathcal{H}}^s} := \|\mathcal{H}^{s/2} f\|_{L^2}$ . Using the Plancherel's identity (4.3.11), we can write

$$\begin{aligned} \|f\|_{H_{\mathcal{H}}^s} &:= \|\mathcal{H}^{s/2} f\|_{L^2} = \left( \|\sigma_{\mathcal{H}}(\xi)^{s/2} \hat{f}(\xi)\|_{HS}^2 \right)^{1/2} = \\ &\left( \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^s \sum_{j=1}^2 |\hat{f}(\xi)_{jj}|^2 \right)^{1/2} = \\ &\left( \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^s \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} f(x) \overline{e_{\xi}^j(x)} dx \right|^2 \right)^{1/2}. \end{aligned} \quad (4.3.14)$$

### 4.3.3 Main results of direct problem

We define the space of the weighted continuous functions  $C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))$  furnished with the maximum norm

$$\|u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))} := \max_{t \in [0, T]} t^{1-\gamma} \|u(t, \cdot)\|_{L^2(\mathbb{R}^2)}. \quad (4.3.15)$$

Here is the main result of the forward problem (4.3.2).

**Theorem 4.3.1.** *Assume that  $f \in C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2)) \cap C_{1-\gamma}([0, T]; H_{\mathcal{H}}^{-1}(\mathbb{R}^2) \cap H_{\mathcal{H}}^1(\mathbb{R}^2))$  and  $u_0 \in L^2(\mathbb{R}^2) \cap H_{\mathcal{H}}^1(\mathbb{R}^2)$ . Then the Cauchy type problem for the time-fractional pseudo-parabolic equation (4.3.2) has a unique solution  $u \in C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))$  such that  $\mathcal{H}u, D_t^{(\alpha, \beta)\mu} u \in C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))$  in the following form:*

$$\begin{aligned}
 u(t, x) &= \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{u}(t, \xi) e_\xi(x)) = \\
 &\sum_{\xi \in \mathbb{N}^2} \left( \hat{u}(t, \xi)_{11} e_\xi^1(x) + \hat{u}(t, \xi)_{22} e_\xi^2(x) \right)
 \end{aligned} \tag{4.3.16}$$

where

$$\begin{aligned}
 \hat{u}(t, \xi)_{mm} &= \hat{u}_0(\xi)_{mm} t^{\gamma-1} E_{\delta, \gamma} \left( -\frac{qB(1+2\xi_2)}{1+aB(1+2\xi_2)} t^\delta \right) + \frac{1}{1+qB(1+2\xi_2)} \times \\
 &\int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta} \left( -\frac{qB(1+2\xi_2)}{1+aB(1+2\xi_2)} (t-\tau)^\delta \right) \hat{f}(\tau, \xi)_{mm} d\tau, \quad m = \overline{1, 2}
 \end{aligned}$$

*Proof of Theorem 4.3.1.* The operator  $\mathcal{H}$  has the symbol (4.3.12), which we can write in the matrix components as

$$\sigma_{\mathcal{H}}(\xi)_{mk} = B(1+2\xi_2)\delta_{mk}, \quad 1 \leq m, k \leq 2,$$

with  $\delta_{mk}$  standing for the Kronecker's delta. Taking the  $\mathcal{H}$ -Fourier transform of (4.3.2) we obtain the collection of Cauchy type problems for matrix-valued Fourier coefficients:

$$D_t^{(\alpha, \beta)\mu} \hat{u}(t, \xi) + (aD_t^{(\alpha, \beta)\mu} + q)\sigma_{\mathcal{H}}(\xi)\hat{u}(t, \xi) = \hat{f}(t, \xi), \quad \xi \in \mathbb{N}^2 \tag{4.3.17}$$

Writing this equation in the matrix form, we see that this is equivalent to the system

$$\begin{aligned}
 D_t^{(\alpha, \beta)\mu} \hat{u}(\xi, t) + \begin{pmatrix} B(1+2\xi_2) & 0 \\ 0 & B(1+2\xi_2) \end{pmatrix} \\
 \times (aD_t^{(\alpha, \beta)\mu} + q)\hat{u}(\xi, t) = \hat{f}(\xi, t), \quad (4.3.18)
 \end{aligned}$$

where  $\xi \in \mathbb{N}^2$  and we put explicitly the diagonal symbol  $\sigma_{\mathcal{H}}(\xi)$ . Rewriting (4.3.17) in terms of matrix coefficients  $\hat{u}(t, \xi) = (\hat{u}(t, \xi)_{mk})_{1 \leq m, k \leq 2}$ , we get the equations

$$D_t^{(\alpha, \beta)\mu} \hat{u}(t, \xi)_{mk} + (aD_t^{(\alpha, \beta)\mu} + q)\sigma_{\mathcal{H}}(\xi)_{mk} \hat{u}(t, \xi)_{mk} = \hat{f}(t, \xi)_{mk}, \quad (4.3.19)$$

for  $\xi \in \mathbb{N}^2$ ,  $1 \leq m, k \leq 2$ .

The main point of our further analysis is that we can make an individual treatment of the equations in (4.3.19). Thus, let us fix  $\xi \in \mathbb{N}^2$  and  $m, k$  with  $1 \leq m, k \leq 2$ . We then study the Cauchy-type problems

$$\begin{cases} D_t^{(\alpha, \beta)\mu} \hat{u}(t, \xi)_{mk} + (aD_t^{(\alpha, \beta)\mu} + q)\sigma_{\mathcal{H}}(\xi)_{mk} \hat{u}(t, \xi)_{mk} = \hat{f}(t, \xi)_{mk}, \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} \hat{u}(t, \xi)_{mk} = \hat{u}_0(\xi)_{mk}, \end{cases} \quad (4.3.20)$$

where  $\xi, m$  are parameters.

Using the result of Lemma 1.3.16, we can write the explicit solution of (4.3.20) in the following form:

$$\begin{aligned} \hat{u}(t, \xi)_{mk} &= \hat{u}_0(\xi)_{mk} t^{\gamma-1} E_{\delta, \gamma} \left( -\frac{qB(1+2\xi_2)}{1+aB(1+2\xi_2)} t^\delta \right) + \frac{1}{1+aB(1+2\xi_2)} \times \\ &\quad \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta} \left( -\frac{qB(1+2\xi_2)}{1+aB(1+2\xi_2)} (t-\tau)^\delta \right) \hat{f}(\tau, \xi)_{mk} d\tau \end{aligned}$$

We apply an inverse Fourier transform to the last result and we have

$$\begin{aligned} u(t, x) &= \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{u}(t, \xi) e_\xi(x)) = \\ &= \sum_{\xi \in \mathbb{N}^2} \left( \hat{u}(t, \xi)_{11} e_\xi^1(x) + \hat{u}(t, \xi)_{22} e_\xi^2(x) \right). \end{aligned} \quad (4.3.21)$$

From now on, for shortly writing we say  $\lambda_\xi = Bq(1+2\xi_2)$  and  $\theta_\xi = (1+aB(1+2\xi_2))^{-1}$ .

Now taking  $L^2$ -norm, for all  $t \in [0, T]$ , we have

$$\begin{aligned}
 & t^{2(1-\gamma)} \|u(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 \leq \\
 & t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \left| \sum_{m=1}^2 \hat{u}_0(\xi)_{mm} \right|^2 |t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta)|^2 + \\
 & + t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \theta_\xi \int_0^t |(t-\tau)^{\delta-1} E_{\delta, \delta}[-\lambda_\xi \theta_\xi (t-\tau)^\delta] \tau^{\gamma-1}| \times \right. \\
 & \left. \times \left| \tau^{1-\gamma} \hat{f}(\tau, \xi)_{mm} \right| d\tau \right)^2 \leq \\
 & \leq M \sum_{\xi \in \mathbb{N}^2} \left( \frac{1}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 + \\
 & + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \max_{t \in [0, T]} \left| t^{1-\gamma} \hat{f}(t, \xi)_{mm} \right| \right)^2 \times \\
 & \times \left( t^{1-\gamma} \theta_\xi \int_0^t |(t-\tau)^{\delta-1} E_{\delta, \delta}[-\lambda_\xi \theta_\xi (t-\tau)^\delta] \tau^{\gamma-1}| d\tau \right)^2 \leq \\
 & M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 + \\
 & + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \max_{t \in [0, T]} \left| t^{1-\gamma} \hat{f}(t, \xi)_{mm} \right| \right)^2 \left( \theta_\xi t^{1-\gamma} \Gamma(\gamma) t^{\delta+\gamma-1} E_{\delta, \delta+\gamma}(-\lambda_\xi \theta_\xi t^\delta) \right)^2 \leq \\
 & M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)|^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} |\theta_\xi|^2 \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \left( \frac{t^\delta}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 \leq \\
 & \leq M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^{-2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \leq \\
 & \leq M \|u_0\|_{L^2(\mathbb{R}^2)}^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \|f(t, \cdot)\|_{H_{\mathcal{H}}^{-1}}^2.
 \end{aligned}$$

Due to the definition of weighted space we have

$$\|u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \leq M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|f\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^{-1})}^2 \right).$$

We have used the following inequality and Plancherel's identity for  $\mathcal{H}$ .

Now, for Landau Hamiltonian, we have

$$\begin{aligned} \mathcal{H}u(t, x) = & \frac{1}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \hat{u}_0(\xi)_{mm} t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta) \lambda_\xi e_\xi(x) + \\ & + \frac{1}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \lambda_\xi \theta_\xi e_\xi(x) \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta}(-\lambda_\xi \theta_\xi (t-\tau)^\delta) \hat{f}(\tau, \xi)_{mm} d\tau. \end{aligned}$$

By taking  $L^2$ -norm, we obtain

$$\begin{aligned} t^{2(1-\gamma)} \|\mathcal{H}u(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 \leq & \frac{t^{2(1-\gamma)}}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 |t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta)|^2 |\lambda_\xi|^2 + \\ & + \frac{t^{2(1-\gamma)}}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\lambda_\xi \theta_\xi|^2 \left( \int_0^t |(t-\tau)^{\delta-1} E_{\delta, \delta}(-\lambda_\xi \theta_\xi (t-\tau)^\delta) \tau^{\gamma-1}| \times \right. \\ & \left. \times \left| \tau^{1-\gamma} \hat{f}(\tau, \xi)_{mm} \right| d\tau \right)^2 \leq \end{aligned}$$

$$\begin{aligned}
 & M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 \lambda_\xi^2 \left( \frac{1}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 \\
 & \quad + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \max_{t \in [0, T]} \left| t^{1-\gamma} \hat{f}(t, \xi)_{mm} \right| \right)^2 \times \\
 & \quad \times \left( t^{1-\gamma} \int_0^t |(t-\tau)^{\delta-1} E_{\delta, \delta} [-\lambda_\xi \theta_\xi (t-\tau)^\delta] \tau^{\gamma-1} d\tau \right)^2 \leq \\
 & \leq M \sum_{\xi \in \mathbb{N}^2} (B(1 + 2\xi_2))^2 \sum_{m=1}^2 |\hat{u}_0(\xi)|^2 + \\
 & M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \max_{t \in [0, T]} \left| t^{1-\gamma} \hat{f}(t, \xi)_{mm} \right| \right)^2 \left( t^{1-\gamma} \Gamma(\gamma) t^{\delta+\gamma-1} E_{\delta, \delta+\gamma} (-\lambda_\xi \theta_\xi t^\delta) \right)^2 \leq \\
 & M \|\mathcal{H}^{1/2} u_0\|_{L^2(\mathbb{R}^2)}^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \left( \frac{t^\delta}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 \leq \\
 & M \|\mathcal{H}^{1/2} u_0\|_{L^2(\mathbb{R}^2)}^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \leq \\
 & \leq \|u_0\|_{H_{\mathcal{H}}^1(\mathbb{R}^2)}^2 + M \max_{t \in [0, T]} t^{2(1-\gamma)} \|f(t, \cdot)\|_{L^2(\mathbb{R}^2)}.
 \end{aligned}$$

According to the formula (4.3.14), if  $u_0(x) \in H_{\mathcal{H}}^1(\mathbb{R}^2)$  and then we can say

$$\sum_{\xi \in \mathbb{N}^2} (B(1 + 2\xi_2))^2 \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 < +\infty.$$

Thus, we have

$$\|\mathcal{H}u(t, x)\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \leq M \left( \|u_0\|_{H_{\mathcal{H}}^1(\mathbb{R}^2)}^2 + \|f\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \right).$$

Now we show the existence of the  $D_t^{(\alpha, \beta)\mu} u(x, t)$  and  $D_t^{(\alpha, \beta)\mu} \mathcal{H}u(x, t)$  which are represented by

$$\begin{aligned}
 D_t^{(\alpha, \beta)\mu} u(x, t) &= \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \hat{u}_0(\xi)_{mm} (-\lambda_\xi \theta_\xi) t^{\gamma-1} E_{\delta, \gamma} (-\lambda_\xi \theta_\xi t^\delta) e_\xi(x) + \\
 &+ \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left[ \hat{f}(t, \xi)_{mm} - \lambda_\xi \theta_\xi^2 \int_0^t (t-\tau)^{\delta-1} E_{\delta, \delta} (-\lambda_\xi \theta_\xi (t-\tau)^\delta) \hat{f}(\tau, \xi)_{mm} d\tau \right] e_\xi(x).
 \end{aligned}$$

and

$$\begin{aligned}
 D_t^{(\alpha,\beta)\mu} \mathcal{H}u(x, t) = & \frac{1}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \hat{u}_0(\xi)_{mm} (-\lambda_\xi^2 \theta_\xi) t^{\gamma-1} E_{\delta,\gamma} (-\lambda_\xi \theta_\xi t^\delta) e_\xi(x) + \\
 & + \frac{1}{q} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left[ \lambda_\xi \hat{f}(t, \xi)_{mm} - \right. \\
 & \left. - \lambda_\xi^2 \theta_\xi^2 \int_0^t (t - \tau)^{\delta-1} E_{\delta,\delta} (-\lambda_\xi \theta_\xi (t - \tau)^\delta) \hat{f}(\tau, \xi)_{mm} d\tau \right] e_\xi(x).
 \end{aligned}$$

respectively.

By taking  $L^2$ -norm, we obtain

$$\begin{aligned}
 t^{2(1-\gamma)} \left\| D_t^{(\alpha,\beta)\mu} u(t, x) \right\|_{L^2(\mathbb{R}^2)}^2 & \leq \\
 & \leq t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mk}|^2 |t^{\gamma-1} E_{\delta,\gamma} (-\lambda_\xi \theta_\xi t^\delta)|^2 |\lambda_\xi \theta_\xi|^2 + \\
 & \quad + t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 + \\
 & \quad + t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\lambda_\xi \theta_\xi|^2 \left( \theta_\xi \int_0^t |(t - \tau)^{\delta-1} E_{\delta,\delta} (-\lambda_\xi \theta_\xi (t - \tau)^\delta) \tau^{\gamma-1}| \times \right. \\
 & \quad \left. \times \left| \tau^{1-\gamma} \hat{f}(\tau, \xi)_{mk} \right| d\tau \right)^2 \leq \\
 & \leq M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 \left( \frac{\lambda_\xi \theta_\xi}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 + t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \\
 & \quad + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \max_{t \in [0, T]} \left| t^{1-\gamma} \hat{f}(t, \xi)_{mm} \right| \right)^2 \left( t^{1-\gamma} \theta_\xi \Gamma(\gamma) t^{\delta+\gamma-1} E_{\delta,\delta+\gamma} (-\lambda_\xi \theta_\xi t^\delta) \right)^2
 \end{aligned}$$

$$\begin{aligned}
 &\leq M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 + t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 + \\
 &\quad \max_{t \in [0, T]} t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^{-2} \sum_{m=1}^2 \left| \hat{f}(t, \xi)_{mm} \right|^2 \leq \\
 &\leq M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|t^{1-\gamma} f(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \max_{t \in [0, T]} t^{2(1-\gamma)} \|f(\tau, \cdot)\|_{H_{\mathcal{H}}^{-1}}^2 d\tau \right).
 \end{aligned}$$

Finally, we get

$$\begin{aligned}
 &\left\| D_t^{(\alpha, \beta)\mu} u(t, x) \right\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \leq \\
 &\quad M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|f\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 + \|f\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^{-1})}^2 \right).
 \end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
 &\left\| D_t^{(\alpha, \beta)\mu} \mathcal{H}u(t, x) \right\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \leq \\
 &\quad M \left( \|u_0\|_{H_{\mathcal{H}}^1}^2 + \|f\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^1)}^2 + \|f\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \right).
 \end{aligned}$$

This completes the proof of existence of the Theorem 4.3.1.  $\square$

*Proof of uniqueness of the solution of the forward problem.* For proving the uniqueness of the result, we suppose that there exist two solutions  $u_1(t, x)$  and  $u_2(t, x)$  of the problem (4.3.2). We denote  $u(t, x) = u_1(t, x) - u_2(t, x)$  function which is solution the problem (4.3.2) in the homogeneous case.

Let us consider the following function

$$u_{\xi}(t) = (u(t, x), e_{\xi}(x))_{L^2(\mathbb{R}^2)}, \quad \xi \in \mathbb{N}^2. \quad (4.3.22)$$

Applying the operator  $D_t^{(\alpha, \beta)\mu}$  to (4.3.22) yields

$$\begin{aligned}
 D_t^{(\alpha, \beta)\mu} u_{\xi}(t) &= \left( D_t^{(\alpha, \beta)\mu} u(t, x), e_{\xi}(x) \right)_{L^2(\mathbb{R}^2)} = \\
 &= (a D_t^{(\alpha, \beta)\mu} + q) (\mathcal{H}u(t, x), e_{\xi}(x))_{L^2(\mathbb{R}^2)}.
 \end{aligned} \quad (4.3.23)$$



Due to the self-adjointness property of the operator  $\mathcal{H}$  and considering the homogeneous condition in (4.3.2), we have

$$\begin{cases} D_t^{(\alpha,\beta)\mu} u_\xi + (aD_t^{(\alpha,\beta)\mu} + q)\lambda_\xi u_\xi = 0, \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} u_\xi = 0 \end{cases} \quad (4.3.24)$$

It is not difficult to show that  $u_\xi = 0$ . Then, from (4.3.22) and the completeness of the system  $\{e_\xi(x)\}_{\xi \in \mathbb{N}^2}$ , we obtain  $u(t, x) \equiv 0$  and this proves the uniqueness of the solution.  $\square$

#### 4.3.4 Inverse source problem

In this section, we are concerned with an inverse source problem of finding a pair of functions  $\{u(t, x), f(x)\}$  for the following fractional pseudo-parabolic equation

$$D_t^{(\alpha,\beta)\mu} u(t, x) + (aD_t^{(\alpha,\beta)\mu} + q)\mathcal{H}u(t, x) = f(x), \quad (t, x) \in (0, T) \times \mathbb{R}^2, \quad (4.3.25)$$

subject to the weighted initial condition

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} u(t, x) = u_0(x), \quad x \in \mathbb{R}^2, \quad (4.3.26)$$

where  $D_t^{(\alpha,\beta)\mu}$  stands for the bi-ordinal Hilfer's fractional derivative in time variable of orders  $0 < \alpha, \beta < 1$  and type  $0 \leq \mu \leq 1$  given in (1.3.4),  $\mathcal{H}$  denotes the Landau Hamiltonian in space variable in the form of (4.3.3).

In order to solve the problem uniquely, we use the following over-determination condition

$$u(T, x) = \psi(x), \quad x \in \mathbb{R}^2, \quad (4.3.27)$$

where  $u_0(x), \psi(x)$  are given functions.

Here we remind the space  $H_{\mathcal{H}}^s$  given with the norm (4.3.14), for any real positive number  $s$  an operator  $\mathcal{H}^{s/2}$ , i.e.,

$$\|g\|_{H_{\mathcal{H}}^s} = \|\mathcal{H}^{s/2} g\|_{L^2(\mathbb{R}^2)}.$$

The main result for this section can be summarized in the following theorem.

**Theorem 4.3.2.** *Assume that  $u_0 \in H_{\mathcal{H}}^1$ ,  $\psi \in H_{\mathcal{H}}^1$ . Then, the problem (4.3.25)-(4.3.27) has a unique pair of solutions  $\{u(t, x), f(x)\}$  determined by*

$$u(t, x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{u}(t, \xi) e_{\xi}(x)), \quad (4.3.28)$$

where

$$\begin{aligned} \hat{u}(t, \xi) = & \hat{u}_0(\xi) t^{\gamma-1} E_{\delta, \gamma} (-\lambda_{\xi} \theta_{\xi} t^{\delta}) + \\ & + t^{\delta} E_{\delta, \delta+1} (-\lambda_{\xi} \theta_{\xi} t^{\delta}) \frac{(\psi, e_{\xi})_{L^2(\mathbb{R}^2)} - (u_0, e_{\xi})_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma} (-\lambda_{\xi} \theta_{\xi} T^{\delta})}{T^{\delta} E_{\delta, \delta+1} (-\lambda_{\xi} \theta_{\xi} T^{\delta})}, \end{aligned}$$

and

$$f(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{f}(\xi) e_{\xi}(x)), \quad (4.3.29)$$

where

$$\hat{f}(\xi) = \frac{q (\mathcal{H}\psi, e_{\xi})_{L^2(\mathbb{R}^2)}}{\lambda_{\xi} \theta_{\xi} T^{\delta} E_{\delta, \delta+1} (-\lambda_{\xi} \theta_{\xi} T^{\delta})} - \frac{q (\mathcal{H}u_0, e_{\xi})_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma} (-\lambda_{\xi} \theta_{\xi} T^{\delta})}{\lambda_{\xi} \theta_{\xi} T^{\delta} E_{\delta, \delta+1} (-\lambda_{\xi} \theta_{\xi} T^{\delta})},$$

here  $\lambda_{\xi} = qB(1 + 2\xi_2)$  and  $\theta_{\xi} = (1 + aB(1 + 2\xi_2))^{-1}$ .

*Existence of solution of the inverse problem.* As we did in the previous section, we expand the functions  $u(t, x)$  and  $f(x)$ , since the system of  $\{e_{\xi}(x)\}_{\xi \in \mathbb{N}^2}$  in the space  $L^2(\mathbb{R}^2)$ :

$$u(t, x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{u}(t, \xi) e_{\xi}(x)), \quad (4.3.30)$$

and

$$f(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} (\hat{f}(\xi) e_{\xi}(x)), \quad (4.3.31)$$

respectively, where

$$\hat{u}(t, \xi) = (u(t, x), e_{\xi}(x))_{L^2(\mathbb{R}^2)}, \quad \xi \in \mathbb{N}^2,$$

$$\hat{f}(\xi) = (f(x), e_\xi(x))_{L^2(\mathbb{R}^2)}, \quad \xi \in \mathbb{N}^2.$$

By substituting the above expressions  $u(t, x)$  and  $f(x)$  into the problem (4.3.25)-(4.3.27) we obtain the following problem for

$$D_t^{(\alpha, \beta)\mu} \hat{u}(t, \xi) + \frac{q\sigma_{\mathcal{H}}}{1 + a\sigma_{\mathcal{H}}(\xi)} \hat{u}(t, \xi) = \frac{\hat{f}(\xi)}{1 + a\sigma_{\mathcal{H}}(\xi)} \quad (4.3.32)$$

$$\lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} \hat{u}(t, \xi) = \hat{u}_0(\xi), \quad \xi \in \mathbb{N}^2, \quad (4.3.33)$$

$$\hat{u}(T, \xi) = \hat{\psi}(\xi), \quad \xi \in \mathbb{N}^2, \quad (4.3.34)$$

and its solution which can be represented by

$$\hat{u}(t, \xi) = \hat{u}_0(\xi) t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta) + \theta_\xi \hat{f}(\xi) t^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi t^\delta) \quad (4.3.35)$$

Using the condition (4.3.34), we find  $\hat{f}(\xi)$

$$\hat{f}(\xi) = \frac{\hat{\psi}(\xi) - \hat{u}_0(\xi) T^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi T^\delta)}{\theta_\xi T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)}.$$

We use the self-adjointness property of the operator  $\mathcal{H}$  and  $\mathcal{H}e_\xi = B(1 + 2\xi_2)e_\xi$ , then we have

$$(u_0, e_\xi)_{L^2(\mathbb{R}^2)} = \frac{q(\mathcal{H}u_0, e_\xi)_{L^2(\mathbb{R}^2)}}{\lambda_\xi} \quad (\psi, e_\xi)_{L^2(\mathbb{R}^2)} = \frac{q(\mathcal{H}\psi, e_\xi)_{L^2(\mathbb{R}^2)}}{\lambda_\xi},$$

where  $\lambda_\xi = qB(1 + 2\xi_2)$ .

Considering above identities we substitute  $\hat{u}(t, \xi)$ ,  $\hat{f}(\xi)$  into the expansions (4.3.30) and (4.3.31) then, we find

$$u(t, x) = \sum_{\xi \in \mathbb{N}^2} Tr(\hat{u}(t, \xi) e_\xi(x)), \quad (4.3.36)$$

where

$$\begin{aligned} \hat{u}(t, \xi) = & \hat{u}_0(\xi) t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta) + \\ & + t^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi t^\delta) \frac{(\psi, e_\xi)_{L^2(\mathbb{R}^2)} - (u_0, e_\xi)_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi T^\delta)}{T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)}, \end{aligned}$$

and

$$f(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} \left( \hat{f}(\xi) e_\xi(x) \right), \quad (4.3.37)$$

where

$$\hat{f}(\xi) = \frac{q(\mathcal{H}\psi, e_\xi)_{L^2(\mathbb{R}^2)}}{\lambda_\xi \theta_\xi T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)} - \frac{q(\mathcal{H}u_0, e_\xi)_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi T^\delta)}{\lambda_\xi \theta_\xi T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)}$$

For the denominators for the  $\hat{u}(t, \xi)$  and  $\hat{f}(\xi)$ , from Lemma 1.4.2, we have

$$\begin{aligned} T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta) &\geq \\ \frac{T^\delta}{\Gamma(\delta+1) + \lambda_\xi \theta_\xi T^\delta} &\geq \frac{T^\delta}{T^\delta + \Gamma(\delta+1) + \lambda_\xi \theta_\xi T^\delta} \geq M_0 > 0. \end{aligned} \quad (4.3.38)$$

It is clearly seen that  $\lambda_\xi \theta_\xi < \infty$  for  $\xi \rightarrow \infty$  and since  $T > T_0 \geq 0$ , the estimate (4.3.38) makes sense.

To prove the existence of the solution we show the convergence of the series of representations of functions  $u(t, x)$ ,  $D_t^{(\alpha, \beta)\mu} u(t, x)$ ,  $\mathcal{H}u(t, x)$  and  $f(x)$ . For that reason, first we calculate  $D_t^{(\alpha, \beta)\mu} u(t, x)$ ,  $\mathcal{H}u(t, x)$ .

$$\begin{aligned} D_t^{(\alpha, \beta)\mu} \hat{u}(t, x) &= (-\lambda_\xi \theta_\xi) \hat{u}_0(\xi) t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta) + \\ &+ E_{\delta, 1}(-\lambda_\xi \theta_\xi t^\delta) \frac{(\psi, e_\xi)_{L^2(\mathbb{R}^2)} - (u_0, e_\xi)_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi T^\delta)}{T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)}, \end{aligned}$$

Now taking  $L^2$ -norm, for all  $t \in [0, T]$ , from Lemma 1.4.1 and Lemma 1.4.2, we have

$$\begin{aligned} t^{2(1-\gamma)} \|u(t, x)\|_{L^2(\mathbb{R}^2)}^2 &\leq t^{2(1-\gamma)} \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 |t^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi t^\delta)|^2 + \\ &+ |t^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi t^\delta)|^2 \left| \frac{(\psi, e_\xi)_{L^2(\mathbb{R}^2)} - (u_0, e_\xi)_{L^2(\mathbb{R}^2)} T^{\gamma-1} E_{\delta, \gamma}(-\lambda_\xi \theta_\xi T^\delta)}{T^\delta E_{\delta, \delta+1}(-\lambda_\xi \theta_\xi T^\delta)} \right|^2 \\ &\leq M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 \left( \frac{1}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 \left( \frac{t^{\delta+1-\gamma}}{1 + \lambda_\xi \theta_\xi t^\delta} \right)^2 \times \\ &\quad \times \left( M_0 \left| (\psi, e_\xi)_{L^2(\mathbb{R}^2)} \right|^2 + M_1 \left| (u_0, e_\xi)_{L^2(\mathbb{R}^2)} \right|^2 \right) \leq \end{aligned}$$

$$M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{u}_0(\xi)_{mm}|^2 + M \sum_{\xi \in \mathbb{N}^2} \sum_{m=1}^2 |\hat{\psi}(\xi)_{mm}|^2 \leq M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\psi\|_{L^2(\mathbb{R}^2)}^2 \right)$$

Here we take into account that

$$T^{\gamma-1} E_{\delta, \gamma} (-\lambda_\xi \theta_\xi T^\delta) \leq \frac{\Gamma(\delta + \gamma)}{\Gamma(\gamma)(\Gamma(\delta + \gamma) + \Gamma(\gamma)\lambda_\xi \theta_\xi T^\delta)} \leq \frac{1}{\Gamma(\gamma)}$$

and  $\frac{1}{M_0 \Gamma(\gamma)} \leq M_1$  which comes from Lemma 1.4.2.

Thus, we have

$$\|u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 \leq M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\psi\|_{L^2(\mathbb{R}^2)}^2 \right).$$

In a similar way one can show that

$$\begin{aligned} \|D_t^{(\alpha, \beta)\mu} u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 &\leq M \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\psi\|_{L^2(\mathbb{R}^2)}^2 \right), \\ \|\mathcal{H}u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 &\leq M \left( \|u_0\|_{H_{\mathcal{H}}^1}^2 + \|\psi\|_{H_{\mathcal{H}}^1}^2 \right), \\ \|D_t^{(\alpha, \beta)\mu} \mathcal{H}u\|_{C_{1-\gamma}([0, T]; L^2(\mathbb{R}^2))}^2 &\leq M \left( \|u_0\|_{H_{\mathcal{H}}^1}^2 + \|\psi\|_{H_{\mathcal{H}}^1}^2 \right). \end{aligned}$$

These complete the proof of existence of the solution.  $\square$

*Uniqueness of solution of the inverse problem.* Uniqueness of the solution of the inverse problem (4.3.25)-(4.3.27) can be proved as usual way represented for the direct problem. Assume  $\{u_1(t, x), f_1(x)\}$  and  $\{u_2(t, x), f_2(x)\}$  be two different pairs of solutions, and let  $u(t, x) = u_1(t, x) - u_2(t, x)$ , also  $f(x) = f_1(x) - f_2(x)$ . Then, it is easy to verify that  $u(t, x)$  satisfy

$$\begin{cases} D_t^{(\alpha, \beta)\mu} u(t, x) + (a D_t^{(\alpha, \beta)\mu} + q) \mathcal{H}u(t, x) = f(x), & (t, x) \in (0, T) \times \mathbb{R}^2, \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} u(t, x) = 0, & x \in \mathbb{R}^2, \\ u(T, x) = 0, & x \in \mathbb{R}^2, \end{cases} \quad (4.3.39)$$

Consider

$$\hat{u}(t, \xi) = (u(t, x), e_\xi(x))_{L^2(\mathbb{R}^2)}, \quad \xi \in \mathbb{N}^2, \quad (4.3.40)$$

and

$$\hat{f}(\xi) = (f(x), e_\xi(x))_{L^2(\mathbb{R}^2)}, \quad \xi \in \mathbb{N}^2. \quad (4.3.41)$$

Taking the fractional derivative and by self-adjointness of the operator  $\mathcal{H}$  and as well as taking into account the homogeneous conditions, we get

$$\left\{ \begin{array}{l} D_t^{(\alpha, \beta)\mu} \hat{u}(t, \xi) + \frac{q\sigma_{\mathcal{H}}}{1 + a\sigma_{\mathcal{H}}(\xi)} \hat{u}(t, \xi) = \frac{\hat{f}(\xi)}{1 + a\sigma_{\mathcal{H}}(\xi)} \\ \lim_{t \rightarrow 0+} I_{0+}^{(1-\mu)(1-\beta)} \hat{u}(t, \xi) = 0, \quad \xi \in \mathbb{N}^2, \\ \hat{u}(T, \xi) = 0, \quad \xi \in \mathbb{N}^2, \end{array} \right. \quad (4.3.42)$$

According to Lemma 1.3.16, from (4.3.42) we obtain

$$\hat{u}(t, \xi) = \frac{\hat{f}(\xi)t^\delta}{1 + aB(1 + 2\xi_2)} E_{\delta, \delta+1} \left( -\frac{qB(1 + 2\xi_2)}{1 + aB(1 + 2\xi_2)} t^\delta \right)$$

From condition  $\hat{u}(T, \xi) = 0$ , one can easily check that  $\hat{u}(t, \xi) = 0$  and  $\hat{f}(\xi) = 0$ . The completeness of the set of function  $\{e_\xi(x)\}$ ,  $\xi \in \mathbb{N}^2$  implies the uniqueness of the solution.  $\square$

### Stability analysis of the inverse source problem

Now we would like to show that the solution of the inverse problem, under certain conditions of Theorem 4.3.2, depends on continuously on the given functions.

**Theorem 4.3.3.** *Let  $\{u(t, x), f(x)\}$ ,  $\{u_p(t, x), f_p(x)\}$  be two solution pairs of the inverse problem corresponding to the data  $(\psi, u_0)$  and its small perturbation  $(\psi_p, u_{0p})$ , respectively. Then the solution of Inverse Problem (4.3.25)-(4.3.27) depends continuously on these data, namely, we have*

$$\|u - u_p\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^1)}^2 \lesssim C \left( \|u_0 - u_{0p}\|_{H_{\mathcal{H}}^1}^2 + \|\psi - \psi_p\|_{H_{\mathcal{H}}^1}^2 \right)$$

and

$$\|f - f_p\|_{H_{\mathcal{H}}^1}^2 \lesssim \left( \|\psi - \psi_p\|_{H_{\mathcal{H}}^2}^2 + \|u_0 - u_{0p}\|_{H_{\mathcal{H}}^2}^2 \right).$$

*Proof.* From the definition of the  $\mathcal{H}$ –Fourier transform

$$(\mathcal{F}_{\mathcal{H}}u(t, \cdot))(\xi) = \hat{u}(t, \xi) = \int_{\mathbb{R}^2} u(t, x) \overline{e_{\xi}(x)} dx$$

and we determine

$$\begin{aligned} \mathcal{F}_{\mathcal{H}}(u(t, \cdot) - u_p(t, \cdot))(\xi) &= \int_{\mathbb{R}^2} (u(t, x) - u_p(t, x)) \overline{e_{\xi}(x)} dx \\ &= \int_{\mathbb{R}^2} u(t, x) \overline{e_{\xi}(x)} dx - \int_{\mathbb{R}^2} u_p(t, x) \overline{e_{\xi}(x)} dx \\ &= \mathcal{F}_{\mathcal{H}}u(t, \cdot)(\xi) - \mathcal{F}_{\mathcal{H}}u_p(t, \cdot)(\xi) = \hat{u}(t, \xi) - \hat{u}_p(t, \xi). \end{aligned}$$

In order to write more concisely the results in Theorem 4.3.2 we assigned the following notations

$$\lambda_{\xi} \theta_{\xi} = Q_{\xi} = \frac{qB(1 + 2\xi_2)}{1 + aB(1 + 2\xi_2)} \quad \text{and} \quad T_{\xi} = \frac{T^{\gamma-1} E_{\delta, \gamma}(-Q_{\xi} T^{\delta})}{T^{\delta} E_{\delta, \delta+1}(-Q_{\xi} T^{\delta})}.$$

One can easily check that  $Q_{\xi} < \infty$  for  $\xi \rightarrow \infty$  and according to Lemma 1.4.2 we have

$$\begin{aligned} |T_{\xi}| &= \left| \frac{T^{\gamma-1} E_{\delta, \gamma}(-Q_{\xi} T^{\delta})}{T^{\delta} E_{\delta, \delta+1}(-Q_{\xi} T^{\delta})} \right| \leq T^{\gamma-\delta-1} \left| \frac{\frac{\Gamma(\delta + \gamma)}{\Gamma(\delta + \gamma) + \Gamma(\gamma) Q_{\xi} T^{\delta}}}{\frac{\Gamma(\delta + 1)}{\Gamma(\delta + 1) + Q_{\xi} T^{\delta}}} \right| = \\ &= \frac{T^{\gamma-\delta-1} (\Gamma(\delta + 1) + Q_{\xi} T^{\delta}) \Gamma(\delta + \gamma)}{(\Gamma(\delta + \gamma) + Q_{\xi} T^{\delta}) \Gamma(\delta + 1)} \leq M^* < \infty \end{aligned}$$

From Plancherel's identity, we have

$$\begin{aligned} \|u(t, \cdot) - u_p(t, \cdot)\|_{H_{\mathcal{H}}^1}^2 &= \\ &= \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} (u(t, x) - u_p(t, x)) \overline{e_{\xi}^j(x)} dx \right|^2 = \\ &= \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 |(\mathcal{F}_{\mathcal{H}}u(t, \cdot)(\xi) - \mathcal{F}_{\mathcal{H}}u_p(t, \cdot)(\xi))|^2 = \end{aligned}$$

$$\begin{aligned}
 & \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 |\hat{u}(t, \xi)_{jj} - \hat{u}_p(t, \xi)_{jj}|^2 \\
 &= \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 \left| \left( \hat{u}_0(\xi)_{jj} t^{\gamma-1} E_{\delta, \gamma} (-Q_\xi t^\delta) + \right. \right. \\
 & \quad \frac{(\psi, e_\xi)_{L^2(\mathbb{R}^2)} t^\delta E_{\delta, \delta+1} (-Q_\xi t^\delta)}{T^\delta E_{\delta, \delta+1} (-Q_\xi T^\delta)} - (u_0, e_\xi)_{L^2(\mathbb{R}^2)} T_\xi t^\delta E_{\delta, \delta+1} (-Q_\xi t^\delta) \\
 & \quad - \hat{u}_{0p}(\xi)_{jj} t^{\gamma-1} E_{\delta, \gamma} (-Q_\xi t^\delta) - \frac{(\psi_p, e_\xi)_{L^2(\mathbb{R}^2)}}{T^\delta E_{\delta, \delta+1} (-Q_\xi T^\delta)} t^\delta E_{\delta, \delta+1} (-Q_\xi t^\delta) + \\
 & \quad \left. \left. + (u_{0p}, e_\xi)_{L^2(\mathbb{R}^2)} T_\xi t^\delta E_{\delta, \delta+1} (-Q_\xi t^\delta) \right) \right|^2 \lesssim \\
 & \lesssim \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 |\hat{u}_0(\xi)_{jj} - \hat{u}_{0p}(\xi)_{jj}|^2 \left( \left( \frac{Mt^{\gamma-1}}{1 + Q_\xi t^\delta} \right)^2 + \left( \frac{MT_\xi t^\delta}{1 + Q_\xi t^\delta} \right)^2 \right) + \\
 & \quad + \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 \left| (\psi - \psi_p, e_\xi)_{L^2(\mathbb{R}^2)} \right|^2 \left( \frac{Mt^\delta}{1 + Q_\xi t^\delta} \right)^2.
 \end{aligned}$$

Considering the weighted space 4.3.15 we get

$$\begin{aligned}
 & \|u(t, \cdot) - u_p(t, \cdot)\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^1)}^2 \lesssim \\
 & \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 |\hat{u}_0(\xi)_{jj} - \hat{u}_{0p}(\xi)_{jj}|^2 + \\
 & + \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 \left| \hat{\psi}_0(\xi)_{jj} - \hat{\psi}_{0p}(\xi)_{jj} \right|^2 \lesssim \\
 & \lesssim \left( \|u_0 - u_{0p}\|_{H_{\mathcal{H}}^1}^2 + \|\psi - \psi_p\|_{H_{\mathcal{H}}^1}^2 \right).
 \end{aligned}$$

Finally, we obtain

$$\|u(t, \cdot) - u_p(t, \cdot)\|_{C_{1-\gamma}([0, T]; H_{\mathcal{H}}^1)}^2 \lesssim \left( \|u_0 - u_{0p}\|_{H_{\mathcal{H}}^1}^2 + \|\psi - \psi_p\|_{H_{\mathcal{H}}^1}^2 \right)$$

From the result in Theorem 4.3.2, writing  $\hat{f}(\xi)$  in the form

$$\hat{f}(\xi) = \left[ \frac{(\mathcal{H}\psi, e_\xi)_{L^2(\mathbb{R}^2)}}{T^\delta E_{\delta, \delta+1} (-Q_\xi T^\delta)} - T_\xi (\mathcal{H}u_0, e_\xi)_{L^2(\mathbb{R}^2)} \right] \frac{q}{\lambda_\xi \theta_\xi}$$



Similarly, one can have a stability result for  $f(x)$  and from Plancherel's identity, we have

$$\begin{aligned}
 \|f(x) - f_p(x)\|_{H_{\mathcal{H}}^1}^2 &= \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 |\hat{f}(\xi) - \hat{f}_p(\xi)|^2 = \\
 &= \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \left| \frac{q}{\lambda_{\xi} \theta_{\xi}} \right|^2 \left[ \frac{(\mathcal{H}\psi, e_{\xi})_{L^2(\mathbb{R}^2)}}{T^{\delta} E_{\delta, \delta+1}(-Q_{\xi} T^{\delta})} - \right. \\
 &\quad \left. -T_{\xi} (\mathcal{H}u_0, e_{\xi})_{L^2(\mathbb{R}^2)} - \frac{(\mathcal{H}\psi_p, e_{\xi})_{L^2(\mathbb{R}^2)}}{T^{\delta} E_{\delta, \delta+1}(-Q_{\xi} T^{\delta})} + T_{\xi} (\mathcal{H}u_{0p}, e_{\xi})_{L^2(\mathbb{R}^2)} \right]^2 \lesssim \\
 &\lesssim \sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^2 \sum_{j=1}^2 \left| \frac{q}{\lambda_{\xi} \theta_{\xi}} \right|^2 \left[ \frac{2 \left| (\mathcal{H}[\psi - \psi_p], e_{\xi})_{L^2(\mathbb{R}^2)} \right|^2}{|T^{\delta} E_{\delta, \delta+1}(-Q_{\xi} T^{\delta})|^2} + \right. \\
 &\quad \left. + 2T_{\xi}^2 \left| (\mathcal{H}[u_0 - u_{0p}], e_{\xi})_{L^2(\mathbb{R}^2)} \right|^2 \right] \lesssim \\
 &\lesssim \sum_{\xi \in \mathbb{N}^2} (B + 2\xi_2 B)^2 \sum_{j=1}^2 \left[ \left| (\mathcal{H}[\psi - \psi_p], e_{\xi})_{L^2(\mathbb{R}^2)} \right|^2 + \right. \\
 &\quad \left. \left| (\mathcal{H}[u_0 - u_{0p}], e_{\xi})_{L^2(\mathbb{R}^2)} \right|^2 \right].
 \end{aligned}$$

Here we considered  $M_0$  defined by (4.3.38). Finally, we have

$$\|f(x) - f_p(x)\|_{H_{\mathcal{H}}^1}^2 \leq N \left( \|\mathcal{H}\psi - \mathcal{H}\psi_p\|_{H_{\mathcal{H}}^1}^2 + \|\mathcal{H}u_0 - \mathcal{H}u_{0p}\|_{H_{\mathcal{H}}^1}^2 \right).$$

Consequently, we have

$$\|f - f_p\|_{H_{\mathcal{H}}^1}^2 \lesssim N \left( \|\psi - \psi_p\|_{H_{\mathcal{H}}^2}^2 + \|u_0 - u_{0p}\|_{H_{\mathcal{H}}^2}^2 \right).$$

It completes the proof. □

## Conclusion of the Chapter 4

In section 4.1 we have analyzed the initial-boundary value problem with a nonlocal  $m$ -point condition in time for the space-degenerate partial differential equation involving the bi-ordinal Hilfer fractional derivative. The main aim of the present investigation is that prove the uniqueness and existence of the result. By using the method of separation variables and considering regularity conditions for sought function we obtain the Legendre ordinary differential equation and the solution of this equation is called Legendre polynomials which form a complete orthogonal system on  $L^2(-1, 1)$  (see [47], p. 38 and [63], p. 511). Uniqueness result of the solution is proved by the completeness properties of the Legendre polynomials. The proof of the existence of the solution is based on Weierstrass M-test. In other words, the necessary conditions for given data are found in order to show that the series which represents the result is uniformly convergent and in this case the properties of Legendre polynomials are used wisely. The solution for the non-local problem is presented explicitly.

The section 4.2 is devoted to studying time-dependent inverse source problem for Langevin-type PDE. The main technique is based on the properties of Fourier-Legendre series. Unique solvability of the inverse source problem is proved and sufficient conditions are found for given data.

In section 4.3, we have considered direct and inverse problems for the pseudo-parabolic equation generated by the Landau Hamiltonian operator  $\mathcal{H}$  and the bi-ordinal Hilfer fractional derivative.

First, we recalled known results of the global Fourier analysis for the Landau Hamiltonian and formulated forward problem. The theorems of uniqueness and existence of the results for forward problem are proved. In this case, the generalized solutions are presented which belong to different spaces according to the order of fractional derivative.

Second, the inverse problem is considered for the 2D Landau Hamiltonian operator  $\mathcal{H}$  and to find the right-hand side of the equation, we have used the value of unknown function at the final time. By using the Fourier analysis for the operator  $\mathcal{H}$  the theorems of uniqueness and existence of for the results of the inverse problem are proved. It is also presented the stability result for the inverse problem.

# Conclusion

The dissertation work is devoted to investigations of direct and inverse problems for the partial differential equations with fractional order integral-differential operators.

In the first Chapter of this dissertation, we have introduced some primary, auxiliary results used in the dissertation. Section 1.1 is devoted to giving brief information on the historical survey of fractional calculus. In section 1.2 some important functional spaces have been introduced. In section 1.3 we give the definitions and useful properties of fractional integral and differential operators such as the Riemann-Liouville, Caputo, right and left-hand-sided bi-ordinal Hilfer fractional differential operator and the regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator. Section 1.4 describes the essential properties of the Mittag-Leffler function.

The main results of the research are as follows:

In the second Chapter, we dealt with investigating the boundary value problem for the mixed equation involving the subdiffusion equation with the hyper-Bessel fractional differential operator and classical wave equation. In section 2.1 we have considered the analogy of the Tricomi problem and proved the uniqueness and existence of the solution. In section 2.2 we have obtained the existence and the uniqueness results for the Frankl-type problem considered for the aforementioned mixed equation. We need to mention that these two problems considered in the same domain and the same equation.

The third Chapter of the present dissertation is devoted to studies of nonlocal problems for the mixed equation involving subdiffusion equation generated by the hyper-Bessel fractional differential equation and fractional wave equation with the bi-ordinal Hilfer fractional derivative. In section 3.1 we presented the theorem about uniqueness and existence of solution of the nonlocal problem for the mixed equation. Section 3.2 differs from previous section with using Bessel operator in the space variable and applying the right-hand-side bi-ordinal Hilfer fractional derivative. We considered the nonlocal problem for mixed equation with the Bessel operator and established the connection between the result of uniqueness and existence and given data. Section 3.3 is devoted to analyzing the problem based on the combination of the elements of the two problems considered in previous sections. Necessary conditions are found for given functions to prove the uniqueness and existence of the solution.

In the fourth chapter, we have considered direct and inverse problems for the pseudo-parabolic equation with fractional order derivative in the time variable. Section 4.1 discusses the  $m$ -point nonlocal problem for the fractional Langevin-type partial differential equation in the case of two variables. The uniqueness and existence results of the nonlocal boundary value problem are presented. Additionally, determining the time-dependent inverse source problem is investigated in section 4.2. With help of the standard procedure of the Fourier method, the theorem of uniqueness and existence of the solution to the inverse source problem is proved.

In section 4.3 we have analyzed the direct and inverse problems for the pseudo-parabolic equation generated by the Landau Hamiltonian operator. The global Fourier analysis developed in [99] played an essential role in considering direct and inverse problem. The necessary conditions for given data are presented to obtain the existence and uniqueness of the generalized solutions for

the direct and inverse problems. In the inverse problem, the value of a sought function at the final time is used as an overdetermination condition. Stability analysis is also developed for the inverse source problem.

Possible future work with studying boundary value problems for mixed equation might be to propose and analyze the problem on two mixed domains connected on the any arbitrary line (not straight) of type changing. Precisely, in chapter 2 and chapter 3, we consider the the conjugation conditions on the line  $t = 0$  and some interval  $x \in \mathcal{I}$  and by using different approach we can develop by taking the gluing conditions on the line of type changing as  $AB = \{(x, t) : t = \phi(x), x \in \mathcal{I}\}$ . In addition, considering the problems aforementioned in this thesis we can solve the problems when the right side of the equation might depend on an unknown function like  $f = f(x, t, u)$ . In this case, we need to use different methods for the unique solvability of the problem, for instance, the homogenization function method, variational approach or other numerical techniques.

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