An L1 type difference / Galerkin spectral scheme for variable-order time-fractional nonlinear diffusion-reaction equations with fixed delay

M. A. Zaky*[a], K. Van Bockstal[b], T. R. Taha[c], D. Suragan[d], A. S. Hendy*[e,f]

[a]*Department of Applied Mathematics, National Research Centre, Dokki, Giza 12622, Egypt*
[b]*Research Group NaM², Department of Electronics and Information systems, Ghent University, Krijgslaan 281, 9000 Ghent, Belgium*
[c]*Department of Computer Science, University of Georgia, Athens, GA 30602-7404, USA*
[d]*Department of Mathematics, Nazarbayev University, Nur-Sultan, Kazakhstan*
[e]*Department of Computational Mathematics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, 19 Mira St., Yekaterinburg 620002, Russia*
[f]*Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt*

Abstract

A linearized spectral Galerkin/finite difference approach is developed for variable fractional-order nonlinear diffusion-reaction equations with a fixed time delay. The temporal discretization for the variable-order fractional derivative is performed by the $L1$-approximation. An appropriate basis function in terms of Legendre polynomials is used to construct the Galerkin spectral method for the spatial discretization of the second-order spatial operator. The main advantage of the proposed approach is that the implementation of the iterative process is avoided for the nonlinear term in the variable fractional-order problem. Convergence and stability estimates for the constructed scheme are proved theoretically by discrete energy estimates. Some numerical experiments are finally provided to demonstrate the efficiency and accuracy of the theoretical findings.

Keywords: variable order diffusion, time delay, L1 difference scheme, Galerkin spectral method, convergence and stability estimates

1. Introduction

Fractional theory of calculus of *variable-order* has recently attracted considerable attention as a powerful mathematical tool for modelling a wide range of discontinuities and nonlinear phenomena [1]. Because many physical systems involve dynamics with memory effects whose behavior changes over time, even while transiting from one fractional-order to another, interest in fractional operators moved increasingly to their variable-order counterparts. The powerful practical implications of these variable-order objects, of course, come at the price of a more complicated mathematical characterization. In contrast to integer-order and fixed fractional-order operators, the variable-order of the fractional operators can be considered as a function of internal or external system variables such as, for example, time, space, state of stress, system energy, temperature, or even a combination of the various variables. Such an extension from fixed-order to variable-order operators also allows the formulation of mathematical models in which the order of the underlying governing equations can be modified using either the system's instantaneous state or its history. As a result, the corresponding model can evolve seamlessly to capture widely dissimilar dynamics without changing the structure of the governing equations which describe the system's response [2]. This emphasizes the evolutionary nature of the variable-order fractional calculus formalism, which indeed can play a critical role in the simulation of nonlinear dynamical models. Recognizing this untapped potential and unique capability of variable-order operators, the scientific community has been intensively investigating applications of variable-order fractional calculus to the modelling of physical and engineering systems [3–9].

*Corresponding author
Email addresses: ma.zaky@yahoo.com (M. A. Zaky*), karel.vanbockstal@ugent.be (K. Van Bockstal), trtaha@uga.edu (T. R. Taha), durvudkhan.suragan@nu.edu.kz (D. Suragan), ahmed.hendy@fsc.bu.edu.eg (A. S. Hendy*)

It is worth noting that viscoelasticity is certainly the playground of the most widespread applications of variable-order fractional calculus since its appearance [10–12]. Although some examples of fractional operators with constant fractional-order successfully mimic experimental test findings, these operators may not always be suited for characterizing viscoelastic behavior. Indeed, nonlinear and complex phenomena occur in several materials, such as biological tissues, polymers, and rubbers. These phenomena imply significant variations in the material's mechanical characteristics that cannot be described by fractional viscoelastic models of constant-orders [2]. Similar problems arise to characterize the mechanical behavior of viscoelastic materials under variable environmental conditions, for instance, changes in temperature or viscoelastic aging effects. As a result, variable-order fractional operators have recently been proposed and utilized in a wide range of applications. For more comprehensive literature review on variable-order fractional operators, associated differential equations and latest applications in natural sciences, we refer the reader to Patnaik et al. [13], Samko [14], Sun et al. [1] and Ortigueira et al. [15].

Time-fractional diffusion equations serve as effective tools for modelling and simulation of various processes with hereditary or memory features, such as anomalous diffusion transport, and hence attract considerable interest [16–19]. Moreover, the theory, application, and implementation of several numerical approaches for the solution of time-fractional differential equations have been proposed in [20–25]. However, the singularity of the solutions of constant-order time-fractional diffusion equations at the beginning of time seems to be physically irrelevant to the subdiffusive transport of the model. The fundamental reason why this phenomenon occurs lies in the incompatibility between the global nature of the power law decaying tails and the locality of the classical initial condition at the beginning of time [26, 27]. It was pointed out in [28] that the regularity of the solution to the variable-order Riemann-Liouville fractional diffusion equation depends on the behavior of the variable order and its derivatives at time $t = 0$, in addition to the standard smoothness assumptions. It was demonstrated that the solution to the problem exhibits full regularity like its integer-order analogue if the variable order has an integer limit at $t = 0$, or it has a singularity at $t = 0$ like in the case of the constant-order time-fractional diffusion equations if the variable order has a non-integer value at time $t = 0$. Accordingly, the variable-order fractional operators can eliminate the nonphysical singularity of the solutions to constant-order time-fractional diffusion equations and produce solutions with full regularity. To the best of our knowledge, up to now, there are no closed form solutions to variable-order fractional diffusion equations. The main reason is that, due to the impact of variable fractional order, analytic techniques such as the Laplace transform cannot be utilized to solve variable-order fractional models. Recently, there are several investigations on numerical schemes to approximate the solutions of variable-order time-fractional diffusion equations in the literature. We mention here the works by Tavares et al. [29], Chen et al. [30], Karniadakis et al. [31–33], Zheng et al. [34–36], Pang and Sun [37], Wei et al. [38, 39] and Liu et al. [40]. Nevertheless, rigorous numerical and mathematical analysis of variable-order fractional differential equations remains wide-open.

Most studies in the literature have been absorbed in the linear variable-order fractional diffusion equations without delay [27, 41, 42]. Although the generalized nonlinear ones are much more useful in real applications, the existing works on the numerical solutions of such problems are quite sparse. Fractional differential equations with delay have recently played a significant role in modelling of many real areas of sciences. A consideration of a method of backward differentiation formula type for solving them is done in [43]. Most recent studies investigated the finite difference/spectral solutions of the fractional diffusion-reaction equations only in implicit schemes [23, 44, 45]. In this work, we develop a linearized explicit finite difference/spectral Galerkin approach for nonlinear variable fractional-order diffusion-reaction equations with a fixed time delay and a drift term, and discuss its stability and the convergence rate. The main advantage of the proposed approach is that the implementation of the iterative process is avoided for the nonlinear term in the variable fractional-order problem. More specifically, we consider the following nonlinear variable order time-fractional reaction–diffusion equations with delay

$$\frac{\partial^{\beta(t)} u}{\partial t^{\beta(t)}} + \lambda \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + f(u(x,t), u(x,t-s)) + g(x,t), \quad 0 < \beta(t) \leq \bar{\beta} < 1,\, x \in \Omega,\, t \in I, \tag{1.1}$$

with the initial-boundary conditions of the form

$$\begin{cases} u(x,t) = \phi(x,t), & x \in \Omega, \quad t \in [-s, 0], \\ u(a,t) = u(b,t) = 0, & t \in I, \end{cases} \tag{1.2}$$

where $I = (0, T] \subset \mathbb{R}$ and $\Omega = (a, b) \subset \mathbb{R}$ are the time and the space domains, respectively. The parameters κ, λ, and s are positive constants. Throughout the paper, we assume that g is a continuous function and f is Lipschitz continuous with the Lipschitz constant L, that is,

$$|f(u_1,v_1)-f(u_2,v_2)| \leq L\left(|u_1-u_2|+|v_1-v_2|\right), \quad \forall u_i, v_i \in \mathbb{R}.$$

In addition, we suppose that ϕ is a smooth function which vanishes at the endpoints of Ω.

The variable-order fractional integral operator ${}_0I_\tau^{\beta(\tau)}$, and the variable-order fractional Caputo operator $\frac{\partial^{\beta(\tau)}}{\partial\tau^{\beta(\tau)}}$ are defined as [46, 47]

$$\begin{aligned} {}_0I_\tau^{\beta(\tau)}u(\tau) &:= \frac{1}{\Gamma\left(\beta(\tau)\right)}\int_0^\tau \frac{u(r)}{(\tau-r)^{1-\beta(\tau)}}dr, \\ \frac{\partial^{\beta(\tau)}}{\partial\tau^{\beta(\tau)}}u(\tau) &:= {}_0I_\tau^{1-\beta(\tau)}u'(\tau) = \frac{1}{\Gamma\left(1-\beta(\tau)\right)}\int_0^\tau \frac{u'(r)}{(\tau-r)^{\beta(\tau)}}dr. \end{aligned} \tag{1.3}$$

The well-posedness of the linear hidden-memory variable–order Caputo time-fractional diffusion equations was discussed in [48, 49], and the space-dependent variable-order fractional Caputo case was discussed in [42]. The well-posedness and the regularity of the solutions to Riemann–Liouville variable-order fractional nonlinear differential equations were discussed in [50, 51]. As a special class of the problem under consideration, an initial boundary value problem for the fractional delayed semilinear diffusion equation with the fractional Laplacian was discussed in [52]. The existence and uniqueness of mild solutions for the abstract time-space evolution equation with delay are investigated by using the semigroup theory of operators and the monotone iterative technique under some quasi-monotone conditions. The maximum principle for multi-term space-time variable-order Riesz-Caputo fractional diffusion equations with nonlinear source function was discussed in [53]. This principle was employed to show the uniqueness of solutions of the variable-order Riesz-Caputo fractional diffusion equations and the continuous dependence of solutions on initial-boundary value conditions. The existence of a solution to nonlinear equations with delay of type (1.1) is a fundamental research question. However, we leave this as an open problem for future work. To the best of our knowledge, we can construct a maximum principle for the problem under consideration by invoking the methodology in [53] and by noticing the existence of a delay in (1.1)-(1.2). This can also be done by invoking the techniques of Barrier analysis in [52]. In the meantime, we investigate the problem's numerical solution and its convergence and stability analysis.

The rest of the paper is arranged as follows. In Section 2, we construct the fully discrete finite difference/Galerkin method for the above model equation. In Section 3, we prove that the fully discrete scheme is unconditionally stable, and the numerical approximation is convergent. In Section 4, we perform two numerical examples to show the accuracy and efficiency of the proposed scheme. Finally, we give the concluding remarks in Section 5.

2. The linearized numerical scheme

Let $\tau = s/N_s$ be the temporal step size, where N_s is a positive integer. Denote $M = \left\lceil \frac{T}{\tau} \right\rceil$. Define $t_k = k\tau$, for each $-N_s \leq n \leq M$ and $u^k = u(\cdot, t_k)$. The $L1$-approximation is defined as [54]

$$\begin{aligned} \left.\frac{\partial^{\beta(t)}f(x,t)}{\partial t^{\beta(t)}}\right|_{t=t_k} &= \int_0^{t_k} f'(x,z)\omega_{1-\beta(t_k)}(t_k-z)dz \\ &= \frac{1}{\Gamma(1-\beta(t_k))}\sum_{q=1}^{k}\frac{f(x,t_q)-f(x,t_{q-1})}{\tau}\int_{t_{q-1}}^{t_q}(t_k-z)^{-\beta(t_k)}dz + r_\tau^k \\ &= \frac{1}{\tau^{\beta(t_k)}\Gamma(2-\beta(t_k))}\sum_{q=1}^{k}a_{k-q}^k\left(f(x,t_q)-f(x,t_{q-1})\right) + r_\tau^k, \end{aligned} \tag{2.4}$$

where $\omega_{\beta(t)}(t) = \frac{t^{\beta(t)-1}}{\Gamma(\beta(t))}$, $t>0$, $a_q^k = (q+1)^{1-\beta(t_k)} - q^{1-\beta(t_k)}$, for each $q \geq 0$. Let $f \in C^2([0,T];L^2(\Omega))$. Then the truncation error r_τ^k fulfills $\|r_\tau^k\| \leq C\tau^{2-\bar{\beta}}$, for each $k = 0, 1, \ldots, M$.

Definition 1. Define [54]

$$D_\tau^{\beta^k} u^k = \frac{\tau^{1-\beta^k}}{\Gamma(2-\beta^k)} \sum_{q=1}^{k} a_{k-q}^k \delta_t u^q = \frac{\tau^{-\beta^k}}{\Gamma(2-\beta^k)} \sum_{q=0}^{k} b_{k-q}^k u^q, \qquad \forall k = 1, \dots, M. \tag{2.5}$$

In this expression, we define $\delta_t u^q = \frac{u^q - u^{q-1}}{\tau}$ and

$$b_0^k = a_0^k, \quad b_k^k = -a_{k-1}^k, \quad b_{k-q}^k = a_{k-q}^k - a_{k-q-1}^k, \text{for } q = 1, \dots, k-1.$$

The coefficients a_q^k satisfy

$$1 = a_0^k > a_1^k > a_2^k > \cdots > a_k^k > 0, \tag{2.6}$$

and

$$(1-\beta^k)(q+1)^{-\beta^k} \le a_q^k \le (1-\beta^k) q^{-\beta^k}.$$

According to (2.6) and [55], the following property is satisfied:

$$\left(D_\tau^{\beta^k} u^k,\ u^k \right) \ge \frac{1}{2} D_\tau^{\beta^k} (u^k)^2. \tag{2.7}$$

The nonlinear source term is discretized and linearized by Taylor's expansion. As a result, we get the following time-discrete system

$$\begin{aligned} &\lambda\, \delta_t u^m + D_\tau^{\beta^m} u^m = \kappa \frac{\partial^2 u^m}{\partial x^2} + f(-u^{m-2} + 2u^{m-1}, u^{m-N_s}) + g^m, \quad 1 \le m \le M, \quad \forall x \in \Omega; \\ &u^m(x) = \phi(x),\ -N_s \le m \le 0,\ x \in \Omega. \end{aligned} \tag{2.8}$$

We here introduce some fundamental properties of Jacobi polynomials. Let N be a positive integer. Denote $P_q^{\gamma,\varsigma}(y)$, γ with $\varsigma > -1$ as the q-th order Jacobi polynomial of indices γ, ς defined on $(-1, 1)$. They satisfy the following three-term-recurrence relation

$$\begin{cases} P_0^{\gamma,\varsigma}(y) = 1, \\ P_1^{\gamma,\varsigma}(y) = \frac{1}{2}(2+\gamma+\varsigma)y + \frac{1}{2}(\gamma - \varsigma), \\ P_{q+1}^{\gamma,\varsigma}(y) = \left(A_q^{\gamma,\varsigma} y - B_q^{\gamma,\varsigma}\right) P_q^{\gamma,\varsigma}(y) - C_q^{\gamma,\varsigma} P_{q-1}^{\gamma,\varsigma}(y), \quad \text{if } 1 \le q \le N, \end{cases}$$

where

$$\begin{cases} A_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+1)(2q+\gamma+\varsigma+2)}{2(q+1)(q+\gamma+\varsigma+1)}, \\ B_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+1)(\varsigma^2-\gamma^2)}{2(q+1)(q+\gamma+\varsigma+1)(2q+\gamma+\varsigma)}, \\ C_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+2)(q+\gamma)(q+\varsigma)}{(q+1)(q+\gamma+\varsigma+1)(2q+\gamma+\varsigma)}. \end{cases}$$

Let $\omega^{\gamma,\varsigma}(y) = (1-y)^\gamma (1+y)^\varsigma$. Then, one has

$$\int_{-1}^{1} P_q^{\gamma,\varsigma}(y) P_j^{\gamma,\varsigma}(y) \omega^{\gamma,\varsigma}(t) dy = h_q^{\gamma,\varsigma} \delta_{q,j}, \qquad \forall q = 0, 1, 2, \dots,$$

where $\delta_{q,j}$ is the Kronecker delta and

$$h_q^{\gamma,\varsigma} = \frac{2^{(\gamma+\varsigma+1)} \Gamma(q+\gamma+1)\Gamma(q+\varsigma+1)}{(2q+\gamma+\varsigma+1) q! \Gamma(q+\gamma+\varsigma+1)}, \qquad \forall q = 0, 1, 2, \dots.$$

For the special case $\gamma = \varsigma = 0$, we get the Legendre polynomials $L_r(t) = P_r^{0,0}(t)$. As a result, we introduce the following space to provide adequate base functions satisfying the boundary conditions [56, 57]

$$\mathcal{W}_N^0 = \text{span}\left\{ \varphi_s(x) : s = 0, 1, \dots, N-2 \right\},$$

where, for each $x \in [a,b]$, the function φ_s is given by

$$\varphi_s(x) = L_s(\hat{x}) - L_{s+2}(\hat{x}) = \frac{2s+3}{2(s+1)}(1-\hat{x}^2)P_s^{1,1}(\hat{x}),$$

where $\hat{x} := \frac{2x-b-a}{b-a} \in [-1,1]$. Next, we introduce the coefficients

$$d^k = \frac{\lambda}{\tau} + \frac{\tau^{-\beta^k}}{\Gamma(2-\beta^k)}, \quad \hat{b}_i^k = \frac{\lambda}{\tau}\delta_{i,k-1} - \frac{\tau^{-\beta^k}}{\Gamma(2-\beta^k)} b_{k-i}^k.$$

Hence, the scheme (2.8) can be rewritten in the following equivalent form:

$$d^m u^m - \kappa \frac{\partial^2 u^m}{\partial x^2} = \sum_{i=0}^{m-1} \hat{b}_i^m u^i + f(-u^{m-2} - 2u^{m-1}, u^{m-N_s}) + g^m, \qquad \forall m = 1, \dots, M.$$

The fully scheme includes the set of approximations $u_N^k \in \mathcal{W}_N^0$, satisfying the system

$$\begin{cases} d^k\left(u_N^k, \upsilon\right) - \kappa\left(\frac{\partial^2}{\partial x^2} u_N^k, \upsilon\right) \\ = \sum_{i=0}^{k-1} \hat{b}_i^k\left(u_N^i, \upsilon\right) + \Big(I_N f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-N_s}), \upsilon\Big) + \left(I_N g^k, \upsilon\right), \quad \forall \upsilon \in \mathcal{W}_N^0, \ \forall k = 1, \dots, M, \\ u_N^k = \pi_N^{1,0}\phi(t_k, x), \quad -N_s \le k \le 0, \end{cases} \tag{2.9}$$

where $\pi_N^{1,0}$ is an appropriate projection operator and I_N is the Lobatto-Gauss type Legendre interpolation operator. The numerical solution can be expanded as

$$u_N^k = \sum_{m=0}^{N-2} \hat{u}_m^k \varphi_m.$$

Lemma 2.1 (Lemma 4.2, [58])**.** *The stiffness matrix S and the mass matrix $\bar{M}$ which invoked later in* (2.10) *are respectively a diagonal matrix and symmetric matrix with the nonzero elements. There elements can be specified as*

$$s_{ii} = 4i + 6, \ i = 0, 1, \dots,$$

$$m_{ij} = m_{ji} = \begin{cases} \frac{b-a}{2j+1} + \frac{b-a}{2j+5}, & i = j, \\ \\ -\frac{b-a}{2j+5}, & i = j+2. \end{cases}$$

Substituting this formula into (2.9) and taking $\upsilon = \varphi_k$, we get the following matrix form representation of the proposed scheme

$$\left(d^k \bar{M} + \kappa S\right) U^k = K^{k-1} + R^{k-1} + G^k, \tag{2.10}$$

where

$$\begin{cases} s_{ij} &= \int_\Omega \varphi_i'(x)\varphi_j'(x)dx, \ S = (s_{ij})_{i,j=0}^{N-2}, \\ m_{ij} &= \int_\Omega \varphi_i(x)\varphi_j(x)dx, \ \bar{M} = (m_{ij})_{i,j=0}^{N-2}, \\ g_i^{k-1} &= \int_\Omega \varphi_i(x)\left(I_N g^k\right)(x)dx, \\ G^k &= (g_0^{k-1}, g_1^{k-1}, \dots, g_{N-2}^{k-1})^\top, \\ h_i^{k-1} &= \int_\Omega \varphi_i(x)\Big(I_N f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-N_s})\Big)(x)dx, \\ R^{k-1} &= (h_0^{k-1}, h_1^{k-1}, \dots, h_{N-2}^{k-1})^\top, \\ U^k &= (\hat{u}_0^k, \hat{u}_1^k, \dots, \hat{u}_{N-2}^k)^\top, \\ K^{k-1} &= -\sum_{j=0}^{k-1} \hat{b}_{k-j}^k \bar{M} U^j. \end{cases}$$

3. Theoretical analysis of the fully discrete scheme

The goal of this section is to investigate the effectiveness of the proposed approach for approximating the solution to problem (1.1)-(1.2). In the first subsection, we discuss the stability analysis and provide the stability theorem. In the second subsection, we discuss the convergence analysis. We begin with introducing some technical lemmas that will be of great importance in the following context.

3.1. Technical lemmas

This section introduces several lemmas that will be used in our analysis [see, Section 3 [58]]. In the sequel, C and C_v will be used to denote generic positive constants that are independent of n, N, and τ and may vary depending on the conditions. We shall also utilize the following notation

$$B(v,f) = \kappa\left(\partial_x v, \partial_x f\right). \tag{3.11}$$

The orthogonal projection operator $\pi_N^{1,0} : H_0^1(\Omega) \to \mathcal{W}_N^0$ satisfies

$$B(v - \pi_N^{1,0} v, f) = 0, \quad v \in H_0^1(\Omega),\ \forall\, f \in \mathcal{W}_N^0.$$

We define the following semi-norms and norms for theoretical purposes [59]:

$$\begin{aligned} |v|_B &:= B(v,v)^{1/2}, \\ \|v\|_B &:= (\|v\|^2 + |v|_B^2)^{1/2}. \end{aligned} \tag{3.12}$$

We recall the following three lemmas from [56].

Lemma 3.1. *Let $\mathcal{L}$ be an arbitrary real number satisfying $\mathcal{L} > 1$. Then, for any function $w \in H_0^1(\Omega) \cap H^{\mathcal{L}}(\Omega)$, the following estimate holds true*

$$|w - \pi_N^{1,0} w|_B \le C N^{1-\mathcal{L}} \|w\|_{H^{\mathcal{L}}}.$$

The following lemma and remark summarize the properties of the interpolation operator I_N, see [44, 56, 58].

Lemma 3.2. *Let $\mathcal{L} \ge 1$. If $w \in H^{\mathcal{L}}(\Omega)$ then*

$$\|w - I_N w\|_{H^q} \le C N^{q-\mathcal{L}} \|w\|_{H^{\mathcal{L}}}, \quad \forall 0 \le q \le 1.$$

Remark 3.1. *A smooth solution to a partial differential equation with fractional order does not always imply a smooth source term, and vice versa. As a result, the regularity order $\mathcal{L}$ of the solution u differs from the regularity order r of the source term g, i.e.*

$$\begin{aligned} \|g - I_N g\| &\le C N^{-r} \|g\|_{H^r}, \quad \forall g \in H^r(\Omega), \\ \|I_N g\| &\le C \|g\|. \end{aligned}$$

The following discrete Grönwall inequality will be needed in our analysis later.

Lemma 3.3. *Assume that $\{u^k | k = 0, 1, \cdots, M\}$ be non-negative sequence, and it satisfies*

$$u^k \le A + B\tau \sum_{s=1}^{k} u^s, \quad k = 0, 1, \cdots, M,$$

then, when $\tau \le \frac{1}{2B}$, it holds that

$$u^k \le A \exp(2Bk\tau), \quad k = 0, 1, \cdots, M,$$

where A and B are non-negative constants.

The following lemma is a direct consequence of Young inequality and inner product properties.

Lemma 3.4. *Suppose that* $\{u_N^k\}_{k=1}^M \in \mathcal{W}_N^0$, *then it holds that*

$$(\delta_t u_N^k, u_N^k) \geq \frac{1}{2\tau}\left(\left\|u_N^k\right\|^2 - \left\|u_N^{k-1}\right\|^2\right).$$

Lemma 3.5. *Suppose that* $\{u_N^k\}_{k=1}^M \in \mathcal{W}_N^0$, *then it holds that*

$$\left(D_\tau^{\beta^k} u_N^k,\ u_N^k\right) + (\delta_t u_N^k,\ u_N^k) \geq \frac{1}{2\tau}\left(\left\|u_N^k\right\|^2 - \left\|u_N^{k-1}\right\|^2\right) + \frac{\tau^{-\beta^k}}{2\Gamma(2-\beta^k)}\left[\left\|u_N^k\right\|^2 - \sum_{i=1}^{k-1}(a_{k-i-1}^k - a_{k-i}^k)\left\|u_N^i\right\|^2 - a_{k-1}^k\left\|u_N^0\right\|^2\right].$$

Proof. A direct consequence of Lemma 3.4 and Eq. (2.7) gives that

$$\left(D_\tau^{\beta^k} u_N^k,\ u_N^k\right) + (\delta_t u_N^k,\ u_N^k) \geq \frac{1}{2\tau}\left(\left\|u_N^k\right\|^2 - \left\|u_N^{k-1}\right\|^2\right) + \frac{1}{2} D_\tau^{\beta^k}\left\|u_N^k\right\|^2.$$

Hence, the result follows from Eq. (2.5). □

Lemma 3.6. *Suppose that* $\{u_N^k\}_{k=1}^M \in \mathcal{W}_N^0$, *then it holds that*

$$\begin{aligned}&\sum_{p=1}^{k}\left(\|u_N^p\|^2 - \left\|u_N^{p-1}\right\|^2\right) + \sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}\left[\|u_N^p\|^2 - \sum_{i=1}^{p-1}(a_{p-i-1}^p - a_{p-i}^p)\left\|u_N^i\right\|^2 - a_{p-1}^p\left\|u_N^0\right\|^2\right]\\ &\geq \left(\left\|u_N^k\right\|^2 - \left\|u_N^0\right\|^2\right) + \frac{1}{T^{\beta^k}\Gamma(1-\beta^k)}\sum_{i=1}^{k}\left\|u_N^i\right\|^2 + \left\|u_N^0\right\|^2\sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}a_{p-1}^p.\end{aligned}$$

Proof. Invoking (2.20) in [60], we similarly have that

$$\begin{aligned}I &:= \sum_{p=1}^{k}\left(\|u_N^p\|^2 - \left\|u_N^{p-1}\right\|^2\right) + \sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}\left[\|u_N^p\|^2 - \sum_{i=1}^{p-1}(a_{p-i-1}^p - a_{p-i}^p)\left\|u_N^i\right\|^2 - a_{p-1}^p\left\|u_N^0\right\|^2\right]\\ &= \sum_{p=1}^{k}\left(\|u_N^p\|^2 - \left\|u_N^{p-1}\right\|^2\right) + \sum_{p=1}^{k}\sum_{i=1}^{p}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}a_{p-i}^p\left(\left\|u_N^i\right\|^2 - \left\|u_N^{i-1}\right\|^2\right)\\ &= \left(\left\|u_N^k\right\|^2 - \left\|u_N^0\right\|^2\right) + \sum_{i=1}^{k}\frac{\tau^{-\beta^k}}{\Gamma(2-\beta^k)}a_{k-i}^k\left\|u_N^i\right\|^2 + \left\|u_N^0\right\|^2\sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}a_{p-1}^p.\end{aligned}$$

As

$$a_0^k > a_1^k > a_2^k > \cdots > a_{k-1}^k \geq (1-\beta^k)(k)^{-\beta^k},$$

we see that

$$I \geq \left(\left\|u_N^k\right\|^2 - \left\|u_N^0\right\|^2\right) + \frac{1}{T^{\beta^k}\Gamma(1-\beta^k)}\sum_{i=1}^{k}\left\|u_N^i\right\|^2 + \left\|u_N^0\right\|^2\sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}a_{p-1}^p.$$

□

3.2. Stability analysis

The weak formulation of the proposed method is as follows: find $\{u_N^m\}_{m=1}^M \in \mathcal{W}_N^0$ such that

$$\begin{aligned}&\left(D_\tau^{\beta^m} u_N^m,\ \upsilon_N\right) + \lambda(\delta_t u_N^m,\ \upsilon_N) + B\left(u_N^m,\ \upsilon_N\right)\\ &\qquad = \left(I_N\, f(2u_N^{m-1} - u_N^{m-2},\ u_N^{m-N_s}),\ \upsilon_N\right) + \left(I_N\, g^m,\ \upsilon_N\right),\ \ \forall \upsilon_N \in \mathcal{W}_N^0,\end{aligned} \tag{3.13}$$

with

$$u_N^m = \pi_N^{1,0}\varphi^m,\ -N_s \le m \le 0.$$

It is a linear iterative approach, which means that we just need to find a solution to a system of linear equations at each time level. The well-known Lax-Milgram lemma proves that the scheme is well-posed.

Now, assume that $\left\{\tilde{u}_N^k\right\}_{k=1}^M$ is the solution to

$$\begin{aligned}&\left(D_\tau^{\beta^m}\tilde{u}_N^m,\ \upsilon_N\right)+\lambda(\delta_t\tilde{u}_N^m,\ \upsilon_N)+B\left(\tilde{u}_N^m,\ \upsilon_N\right)\\&\qquad=\left(I_N\,f(2\tilde{u}_N^{m-1}-\tilde{u}_N^{m-2},\tilde{u}_N^{m-N_s}),\ \upsilon_N\right)+\left(I_N\,\tilde{g}^m,\ \upsilon_N\right),\ \ \forall\upsilon_N\in\mathcal{W}_N^0,\end{aligned}\tag{3.14}$$

with initial conditions

$$\tilde{u}_N^m = \pi_N^{1,0}\varphi^m,\ \ -N_s \le m \le 0.$$

In the following theorem, we provide the stability with respect to the source function g.

Theorem 3.1. *Assume that $\lambda > 0$, $1 \le m \le M$. Then, the fully discrete scheme (3.13) is unconditional stable, which means that for all $\tau > 0$, it holds that*

$$\left\|u_N^m-\tilde{u}_N^m\right\|^2 \le \frac{C\tilde{C}T}{\lambda}\sum_{s=1}^m\|g^s-\tilde{g}^s\|^2,$$

where $\tilde{C}=\exp\left(\frac{4}{\lambda}\left(1+7C^2L^2\right)T\right)$ is a positive constant, in which C is a generic positive constant independent on τ and N.

Proof. Denote $\eta_N^k = u_N^k-\tilde{u}_N^k$. Subtracting (3.14) from (3.13), it holds that

$$\begin{aligned}&\left(D_\tau^{\beta^k}\eta_N^k,\ \upsilon_N\right)+\lambda(\delta_t\eta_N^k,\ \upsilon_N)+B\left(\eta_N^k,\ \upsilon_N\right)\\&=\left(I_N\,f\left(2u_N^{k-1}-u_N^{k-2},\ u_N^{k-N_s}\right)-I_N\,f\left(2\tilde{u}_N^{k-1}-\tilde{u}_N^{k-2},\tilde{u}_N^{k-N_s}\right),\ \upsilon_N\right)+\left(I_N\,g^k-I_N\tilde{g}^k,\ \upsilon_N\right).\end{aligned}\tag{3.15}$$

Using the ϵ-Young inequality and the Minkowski inequality side by side to the Lipschitz condition on f, we derive the following by the aid of Remark 3.1,

$$\begin{aligned}&\left(I_N\,f(2u_N^{k-1}-u_N^{k-2},\ u_N^{k-N_s})-I_N\,f(2\tilde{u}_N^{k-1}-\tilde{u}_N^{k-2},\tilde{u}_N^{k-N_s}),\ \upsilon_N\right)\\&\le\left\|I_N\ \left(f(2u_N^{k-1}-u_N^{k-2},\ u_N^{k-N_s})-\ f(2\tilde{u}_N^{k-1}-\tilde{u}_N^{k-2},\tilde{u}_N^{k-N_s})\right)\right\|\|\upsilon_N\|\\&\le\frac{\epsilon}{2}\left\|I_N\ \left(f(2u_N^{k-1}-u_N^{k-2},\ u_N^{k-N_s})-\ f(2\tilde{u}_N^{k-1}-\tilde{u}_N^{k-2},\tilde{u}_N^{k-N_s})\right)\right\|^2+\frac{1}{2\epsilon}\left\|\upsilon_N\right\|^2\\&\le\frac{\epsilon}{2}C^2L^2\left(\|2\eta_N^{k-1}-\eta_N^{k-2}\|+\|\eta_N^{k-N_s}\|\right)^2+\frac{1}{2\epsilon}\left\|\upsilon_N\right\|^2\\&\le\epsilon C^2L^2\|2\eta_N^{k-1}-\eta_N^{k-2}\|^2+\epsilon C^2L^2\|\eta_N^{k-N_s}\|^2+\frac{1}{2\epsilon}\|\upsilon_N\|^2\end{aligned}$$

and

$$\left(I_N\,g^k-I_N\,\tilde{g}^k,\ \upsilon_N\right)\le\frac{\epsilon}{2}C\left\|g^k-\tilde{g}^k\right\|^2+\frac{1}{2\epsilon}\|\upsilon_N\|^2.$$

Then, (3.15) becomes

$$\begin{aligned}&(D_\tau^{\beta^k}\eta_N^k,\ \upsilon_N)+\lambda(\delta_t\eta_N^k,\ \upsilon_N)+B(\eta_N^k,\ \upsilon_N)\\&\le\frac{1}{\epsilon}\|\upsilon_N\|^2+4\epsilon C^2L^2\|\eta_N^{k-1}\|^2+2\epsilon C^2L^2\|\eta_N^{k-2}\|^2+\epsilon C^2L^2\|\eta_N^{k-N_s}\|^2+\frac{\epsilon C}{2}\|g^k-\tilde{g}^k\|^2,\end{aligned}$$

Invoking Lemma 3.4, Lemma 3.5 and taking $\upsilon_N = \eta_N^k$ yield

$$\left(\left\|\eta_N^k\right\|^2 - \left\|\eta_N^{k-1}\right\|^2\right) + \frac{\tau^{1-\beta^k}}{\lambda\Gamma(2-\beta^k)}\left[\left\|\eta_N^k\right\|^2 - \sum_{i=1}^{k-1}(a_{k-i-1}^k - a_{k-i}^k)\left\|\eta_N^i\right\|^2 - a_{k-1}^k\left\|\eta_N^0\right\|^2\right] + \frac{2\tau}{\lambda}|\eta_N^k|_B^2$$
$$\leq \frac{2\tau}{\lambda\epsilon}\|\eta_N^k\|^2 + \frac{8\epsilon\tau C^2L^2}{\lambda}\|\eta_N^{k-1}\|^2 + \frac{4\epsilon\tau C^2L^2}{\lambda}\|\eta_N^{k-2}\|^2 + \frac{2\epsilon\tau C^2L^2}{\lambda}\|\eta_N^{k-N_s}\|^2 + \frac{\tau\epsilon C}{\lambda}\|g^k - \tilde{g}^k\|^2. \quad (3.16)$$

Eliminating the positive term $|\eta_N^k|_B^2$, replacing k by p in (3.16) and summing the result up for p from 1 to k give

$$\sum_{p=1}^{k}\left(\|\eta_N^p\|^2 - \left\|\eta_N^{p-1}\right\|^2\right) + \sum_{p=1}^{k}\frac{\tau^{1-\beta^p}}{\lambda\Gamma(2-\beta^p)}\left[\|\eta_N^p\|^2 - \sum_{i=1}^{p-1}(a_{p-i-1}^p - a_{p-i}^p)\left\|\eta_N^i\right\|^2 - a_{p-1}^p\left\|\eta_N^0\right\|^2\right]$$
$$\leq \left(\frac{2\tau}{\lambda\epsilon} + \frac{14\epsilon\tau C^2L^2}{\lambda}\right)\sum_{p=1}^{k}\|\eta_N^p\|^2 + \frac{\tau\epsilon C}{\lambda}\sum_{p=1}^{k}\|g^p - \tilde{g}^p\|^2.$$

Let $\epsilon = 1$. By the aid of Lemma 3.6, we get that

$$\left(\left\|\eta_N^k\right\|^2 - \left\|\eta_N^0\right\|^2\right) + \frac{1}{T^{\beta^k}\Gamma(1-\beta^k)}\sum_{i=1}^{k}\left\|\eta_N^i\right\|^2 + \left\|\eta_N^0\right\|^2\sum_{p=1}^{k}\frac{\tau^{-\beta^p}}{\Gamma(2-\beta^p)}a_{p-1}^p$$
$$\leq \left(\frac{2\tau}{\lambda} + \frac{14\tau C^2L^2}{\lambda}\right)\sum_{p=1}^{k}\|\eta_N^p\|^2 + \frac{\tau C}{\lambda}\sum_{p=1}^{k}\|g^p - \tilde{g}^p\|^2. \quad (3.17)$$

As $\eta_N^0 = 0$ and due to the positivity of the second term of the l.h.s of (3.17), we finally get that

$$\left\|\eta_N^k\right\|^2 \leq \frac{2}{\lambda}\left(1 + 7C^2L^2\right)\tau\sum_{p=1}^{k}\|\eta_N^p\|^2 + \frac{CT}{\lambda}\sum_{p=1}^{k}\|g^p - \tilde{g}^p\|^2.$$

An application of the Grönwall inequality (see Lemma 3.3) yields for $\tau \leq \frac{4}{\lambda}\left(1 + 7C^2L^2\right)$ that

$$\left\|\eta_N^k\right\|^2 \leq \frac{C\tilde{C}T}{\lambda}\sum_{p=1}^{k}\|g^p - \tilde{g}^p\|^2, \quad \tilde{C} = \exp\left(\frac{4}{\lambda}\left(1 + 7C^2L^2\right)T\right).$$

As a result, the scheme is unconditionally stable. □

3.3. *Convergence analysis*

In this subsection, we use error estimates to analyze the convergence of the fully discrete scheme (3.13).

Theorem 3.2. *Assume that $\lambda > 0$. Let u^m be the exact solution to* (1.1) *at time t_m for $m = -N_s, \ldots, M$, and let $\{u_N^m\}_{m=-N_s}^M$ be the solution to* (3.13). *Assume that $u \in C^2\left([0,T]; L^2(\Omega)\right) \cap C^1\left([0,T]; H^{\mathcal{L}}(\Omega)\right)$ and $g \in C\left((0,T); H^r(\Omega)\right)$. Then, it holds that*

$$\|u^m - u_N^m\| \leq C(N^{1-\mathcal{L}} + N^{-r} + \tau)\,,\ 1 \leq m \leq M,$$

where C is a suitable positive constant independent of N and τ.

Proof. Let $u^m - u_N^m = e_N^m = (u^m - \pi_N^{1,0}u^m) + (\pi_N^{1,0}u^m - u_N^m) \triangleq \tilde{e}_N^m + \hat{e}_N^m$. The weak formulation of equation (1.1) at time t_m is given by

$$({}_0^C D_t^{\beta^m} u^m,\ \upsilon_N) + \lambda(\partial_t u^m, \upsilon_N) + B\left(u^m,\ \upsilon_N\right) = (f(u^m, u^{m-N_s}), \upsilon_N) + (g^m,\ \upsilon_N). \quad (3.18)$$

Subtracting (3.13) from (3.18), then the error equation satisfies

$$(D_\tau^{\beta^m}\hat{e}_N^m,\ \upsilon_N)+\lambda(\delta_t\hat{e}_N^m,\upsilon_N)+B(\hat{e}_N,\upsilon_N)\triangleq\Upsilon_1^m+\Upsilon_2^m+\Upsilon_3^m+\Upsilon_4^m, \tag{3.19}$$

where

$$\begin{aligned}
\Upsilon_1^m&=\Big(I_Nf(u^m,\ u^{m-N_s})-I_Nf(2u_N^{m-1}-u_N^{m-2},\ u_N^{m-N_s}),\ \upsilon_N\Big),\\
\Upsilon_2^m&=\big(f(u^m,\ u^{m-N_s})-I_Nf(u^m,\ u^{m-N_s}),\ \upsilon_N\big),\\
\Upsilon_3^m&=\Big((D_\tau^{\beta^m}+\lambda\delta_t)\pi_N^{1,0}u^m-({}_0^CD_t^{\beta^m}+\lambda\partial_t)u^m,\ \upsilon_N\Big),\\
\Upsilon_4^m&=(g^m-I_Ng^m,\upsilon_N).
\end{aligned}$$

We next estimate the right-hand terms Υ_1^m, Υ_2^m, Υ_3^m and Υ_4^m. For the first term Υ_1^m, we have that

$$\begin{aligned}
\Upsilon_1^m&=(I_Nf(u^m,\ u^{m-N_s})-I_Nf(2u^{m-1}-u^{m-2},\ u^{m-N_s}),\ \upsilon_N)\\
&\quad+(I_Nf(2u^{m-1}-u^{m-2},\ u^{m-N_s})-I_Nf(2u_N^{m-1}-u_N^{m-2},\ u_N^{m-N_s}),\ \upsilon_N)\\
&\triangleq\Upsilon_{11}^m+\Upsilon_{12}^m.
\end{aligned} \tag{3.20}$$

Applying Taylor expansion, it holds that

$$\begin{aligned}
f(u^m,\ u^{m-N_s})&=f(2u^{m-1}-u^{m-2},\ u^{m-N_s})+(u^m-2u^{m-1}+u^{m-2})\ f_1'(\xi,\ u^{m-N_s})\\
&=f(2u^{m-1}-u^{m-2},\ u^{m-N_s})+\tilde{c}_u\tau^2.
\end{aligned}$$

Moreover, by means of Hölder's inequality and Young's inequality, we have that

$$\begin{aligned}
\Upsilon_{11}^m&\le\|I_Nf(u^m,\ u^{m-N_s})-I_Nf(-u^{m-2}+2u^{m-1},\ u^{m-n})\|\ \|\upsilon_N\|\\
&\le C\|f(u^m,\ u^{m-N_s})-f(-u^{m-2}+2u^{m-1},\ u^{m-N_s})\|\ \|\upsilon_N\|\\
&\le\frac{\epsilon}{2}\tilde{c}_u\tau^4+\frac{1}{2\epsilon}\|\upsilon_N\|^2,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\Upsilon_{12}^m&\le LC(\|2e_N^{m-1}-e_N^{m-2}\|+\|e_N^{m-N_s}\|)\|\upsilon_N\|\\
&\le LC(\|2\hat{e}_N^{m-1}-\hat{e}_N^{m-2}\|+\|\hat{e}_N^{m-N_s}\|+\|2\tilde{e}_N^{m-1}-\tilde{e}_N^{m-2}\|+\|\tilde{e}_N^{m-N_s}\|)\|\upsilon_N\|\\
&\le\frac{8\epsilon}{2}CL^2\|\hat{e}_N^{m-1}\|^2+\frac{2\epsilon}{2}CL^2\|\hat{e}_N^{m-2}\|^2+\frac{\epsilon}{2}L^2\|\hat{e}_N^{m-N_s}\|^2+\frac{8\epsilon}{2}CL^2\|\tilde{e}_N^{m-1}\|^2\\
&\quad+\frac{2\epsilon}{2}CL^2\|\tilde{e}_N^{m-2}\|^2+\frac{\epsilon}{2}CL^2\|\tilde{e}_N^{m-N_s}\|^2+\frac{1}{2\epsilon}\|\upsilon_N\|^2.
\end{aligned} \tag{3.22}$$

Additionally, owing to Lemma 3.1, we notice that

$$\|\tilde{e}_N^{m-1}\|^2\le\frac{C}{C_1}N^{2-2\mathcal{L}}\|u^{m-1}\|_{\mathcal{L}}^2,\ \ \|\tilde{e}_N^{m-2}\|^2\le\frac{C}{C_1}N^{2-2\mathcal{L}}\|u^{m-2}\|_{\mathcal{L}}^2,\ \ \|\tilde{e}_N^{m-N_s}\|^2\le\frac{C}{C_1}N^{2-2\mathcal{L}}\|u^{m-N_s}\|_{\mathcal{L}}^2.$$

Then, (3.22) becomes

$$\Upsilon_{12}^m\ \le\ 4\epsilon CL^2\|\hat{e}_N^{m-1}\|^2+\epsilon CL^2\|\hat{e}_N^{m-2}\|^2+\frac{\epsilon}{2}CL^2\|\hat{e}_N^{m-N_s}\|^2+\frac{C}{C_1}N^{2-2\mathcal{L}}\|u\|_{\mathcal{L}}^2+\frac{1}{2\epsilon}\|\upsilon_N\|^2. \tag{3.23}$$

Substituting (3.21) and (3.23) into (3.20), we can derive that

$$\Upsilon_1^m\ \le\ \frac{1}{\epsilon}\|\upsilon_N\|^2+4\epsilon CL^2\|\hat{e}_N^{m-1}\|^2+\epsilon CL^2\|\hat{e}_N^{m-2}\|^2+\frac{\epsilon}{2}CL^2\|\hat{e}_N^{m-N_s}\|^2+\frac{C}{C_1}N^{2-2\mathcal{L}}\|u\|_{\mathcal{L}}^2+\frac{\epsilon}{2}\tilde{c}_u\tau^2. \tag{3.24}$$

For the second term Υ_2^m, by means of Hölder's inequality, we see that

$$\Upsilon_2^m \leq \frac{\epsilon}{2} C N^{-2r} \|u\|_r^2 + \frac{1}{2\epsilon} \|\upsilon_N\|^2 . \tag{3.25}$$

For the third term Υ_3^m, we have that

$$\begin{aligned}\Upsilon_3^m &= ((D_\tau^{\beta^m} + \lambda\delta_t)\pi_N^{1,0} u^m - ({}_0^C D_t^{\beta^m} + \lambda\partial_t)\pi_N^{1,0} u^m, \upsilon_N) + (({}_0^C D_t^{\beta^m} + \lambda\partial_t)\pi_N^{1,0} u^m - ({}_0^C D_t^{\beta^m} + \lambda\partial_t) u^m, \upsilon_N) \\ &= (\pi_N^{1,0}((D_\tau^{\beta^m} + \lambda\delta_t) u^m - ({}_0^C D_t^{\beta^m} + \lambda\partial_t) u^m), \upsilon_N) - (({}_0^C D_t^{\beta^m} + \lambda\partial_t)\tilde{e}_N^m,\ \upsilon_N) \\ &\triangleq \Upsilon_{31}^m + \Upsilon_{32}^m.\end{aligned} \tag{3.26}$$

Using (2.4) and Hölder's inequality, we obtain that

$$\begin{aligned}\Upsilon_{31}^m &\leq \frac{\epsilon}{2}\|\pi_N^{1,0}((D_\tau^{\beta^m} + \lambda\delta_t) u^m - ({}_0^C D_t^{\beta^m} + \lambda\partial_t) u^m)\|^2 + \frac{1}{2\epsilon}\|\upsilon_N\|^2 \\ &\leq \frac{\epsilon}{2} C \|((D_\tau^{\beta^m} + \lambda\delta_t) u^m - ({}_0^C D_t^{\beta^m} + \lambda\partial_t) u^m)\|^2 + \frac{1}{2\epsilon}\|\upsilon_N\|^2 \\ &\leq \frac{\epsilon}{2} C_{1,u}\tau^2 + \frac{1}{2\epsilon}\|\upsilon_N\|^2.\end{aligned}$$

Furthermore, according to Lemma 3.1, we have that

$$\begin{aligned}\Upsilon_{32}^m &\leq \frac{\epsilon}{2} C N^{2-2\mathcal{L}} \|{}_0^C D_t^{\beta^m} u^m\|_{\mathcal{L}}^2 + \frac{1}{2\epsilon}\|\upsilon_N\|^2 \\ &\leq \frac{\epsilon}{2} C N^{2-2\mathcal{L}} \|({}_0^C D_t^{\beta^m} + \lambda\partial_t) u\|_{\mathcal{L}}^2 + \frac{1}{2\epsilon}\|\upsilon_N\|^2.\end{aligned}$$

Thus (3.26) becomes

$$\Upsilon_3^m \leq \frac{\epsilon}{2} C N^{2-2\mathcal{L}} \|({}_0^C D_t^{\beta^m} + \lambda\partial_t) u\|_{\mathcal{L}}^2 + \frac{\epsilon}{2} C_{2,u}\tau^2 + \frac{1}{\epsilon}\|\upsilon_N\|^2. \tag{3.27}$$

For the fourth term Υ_4^m, by invoking Remark 3.1, it holds that

$$\Upsilon_4^m \leq \frac{\epsilon}{2} C N^{-2r} \|u\|_r^2 + \frac{1}{2\epsilon} \|\upsilon_N\|^2 . \tag{3.28}$$

Substituting (3.24), (3.25), (3.27) and (3.28) into (3.19), we can infer that

$$\begin{aligned}&(D_\tau^{\beta^m} \hat{e}_N^m,\ \upsilon_N) + \lambda(\delta_t \hat{e}_N^m, \upsilon_N) + B(\hat{e}_N, \upsilon_N) \\ &\leq \frac{5}{2\epsilon}\|\upsilon_N\|^2 + 4\epsilon C L^2 \|\hat{e}_N^{m-1}\|^2 + \epsilon C L^2 \|\hat{e}_N^{m-2}\|^2 + \frac{\epsilon}{2} C L^2 \|\hat{e}_N^{m-N_s}\|^2 + \tilde{\mathcal{G}},\end{aligned} \tag{3.29}$$

where

$$\tilde{\mathcal{G}} = \epsilon \tilde{C} N^{2-2\mathcal{L}} \left(\|({}_0^C D_t^{\beta^m} + \lambda\partial_t) u\|_{\mathcal{L}}^2 + \|u\|_{\mathcal{L}}^2 \right) + \epsilon \tilde{C} N^{-2r} \|u\|_r^2 + \epsilon \tilde{C}_u \tau^2.$$

Taking $\upsilon_N = \hat{e}_N^m$ in (3.29) and applying (3.12), we can conclude that

$$(D_\tau^{\beta^m} \hat{e}_N^m, \hat{e}_N^m) + \lambda(\delta_t \hat{e}_N^m, \hat{e}_N^m) + |\hat{e}_N^m|^2 \leq \frac{5}{2\epsilon}\|\hat{e}_N^m\|^2 + 4\epsilon C L^2 \|\hat{e}_N^{m-1}\|^2 + \epsilon C L^2 \|\hat{e}_N^{m-2}\|^2 + \frac{\epsilon}{2} C L^2 \|\hat{e}_N^{m-N_s}\|^2 + \tilde{\mathcal{G}}.$$

Now, we can use the same methodology as in Theorem 3.1 to complete the proof. □

4. Numerical verification

In this section, two numerical experiments are considered to demonstrate the effectiveness and convergence orders of the scheme.

Table 1: The rate of convergence and the errors versus τ and σ with $N = 50$ for example 1.

τ	$\sigma = 2$		$\sigma = 1.5$		$\sigma = 1.1$	
	Error	Order	Error	Order	Error	Order
1/100	6.585×10^{-4}	$--$	1.882×10^{-4}	$--$	2.509×10^{-5}	$--$
1/200	3.209×10^{-4}	1.037	2.537×10^{-5}	1.025	1.237×10^{-5}	1.020
1/400	1.570×10^{-4}	1.031	4.566×10^{-5}	1.019	6.122×10^{-6}	1.015
1/800	7.715×10^{-5}	1.025	2.260×10^{-5}	1.014	3.035×10^{-6}	1.012
1/1600	3.802×10^{-5}	1.021	1.121×10^{-5}	1.010	1.508×10^{-6}	1.009

Example 1. *We consider the following nonlinear delay reaction-diffusion problem*

$$\begin{aligned}
&\frac{\partial^{\beta(t)} u}{\partial t^{\beta(t)}}(x,t) + \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - 2u(x,t) + \frac{u(x,t-0.1)}{1+u^2(x,t-0.1)} + g(x,t), \quad x \in (0,1), \quad t \in (0,1],\\
&u(0,t) = u(1,t) = 0, \quad t \in (0,1), \qquad (4.30)\\
&u(x,t) = (1+t)^\sigma \sin(\pi x), \quad x \in \Omega,\, t \in [-0.1, 0], \quad \beta(t) = \frac{2+\sin(t)}{4},
\end{aligned}$$

where $g(x,t)$ is a given function such that problem (4.30) *has the exact solution $u(x,t) = (1+t)^\sigma \sin(\pi x)$.*

According to the assumptions on the solution u in Theorem 3.2, We can testify the numerical experiments at $\sigma = 2$ but additionally we also run some numerical experiments for different values of σ not exceeds 2 for which the conditions of Theorem 3.2 cannot be satisfied. So, we devote Table 1 to show the L^2-errors and corresponding convergence orders for $\sigma = 2,\ 1.5,\ 1.1$ with $N = 50$. These findings support the theoretically obtained convergence order in time that was obtained in Theorem 3.2. Figure 1 shows the spatial convergence orders for various values of σ at $\tau = 1/1600$. Additionally, Figure 2 shows the space-time error functions with $\sigma = 1.1,\ 1.5,\ 2,\ N = 50$ and $\tau = 1/2000$.

Example 2. *We investigate the following variable-order fractional equation, where the dynamics of the solution are intriguing and exact solution is unknown*

$$\frac{\partial^{\beta(t)} u}{\partial t^{\beta(t)}}(x,t) + \lambda \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)(1-u(x,t))(1+u(x,t-1.5)), \quad x \in (-15,15), \quad t \in (0,1], \quad (4.31)$$

with the initial value $u(x,0) = e^{-2x^2}$. The variable orders $\beta(t)$ is given by

$$\beta(t) = \beta(T) + (\beta(0) - \beta(T)) \left(1 - \frac{t}{T} - \frac{\sin\left(2\pi\left(1 - \frac{t}{T}\right)\right)}{2\pi}\right).$$

For $N = 100$ and $\tau = 0.003$, the behavior of the numerical solution to problem (4.31) is displayed in Figure 3 with $\lambda = 0$ and in Figure 4 with $\lambda = 1$ for the three cases:

(I) $\beta(0) = 0.0$ and $\beta(1) = 0.9$.

(II) $\beta(0) = 0.5$ and $\beta(1) = 0.9$.

(III) $\beta(0) = 0.8$ and $\beta(1) = 0.9$.

For $\lambda = 0$, we see that the solution for case (I) has initial weak singularities near $t = 0$, whereas the solution for cases (II) and (II) is smooth near $t = 0$. For $\lambda = 1$, these observations do not appear, and the solution is smooth for all considered cases.

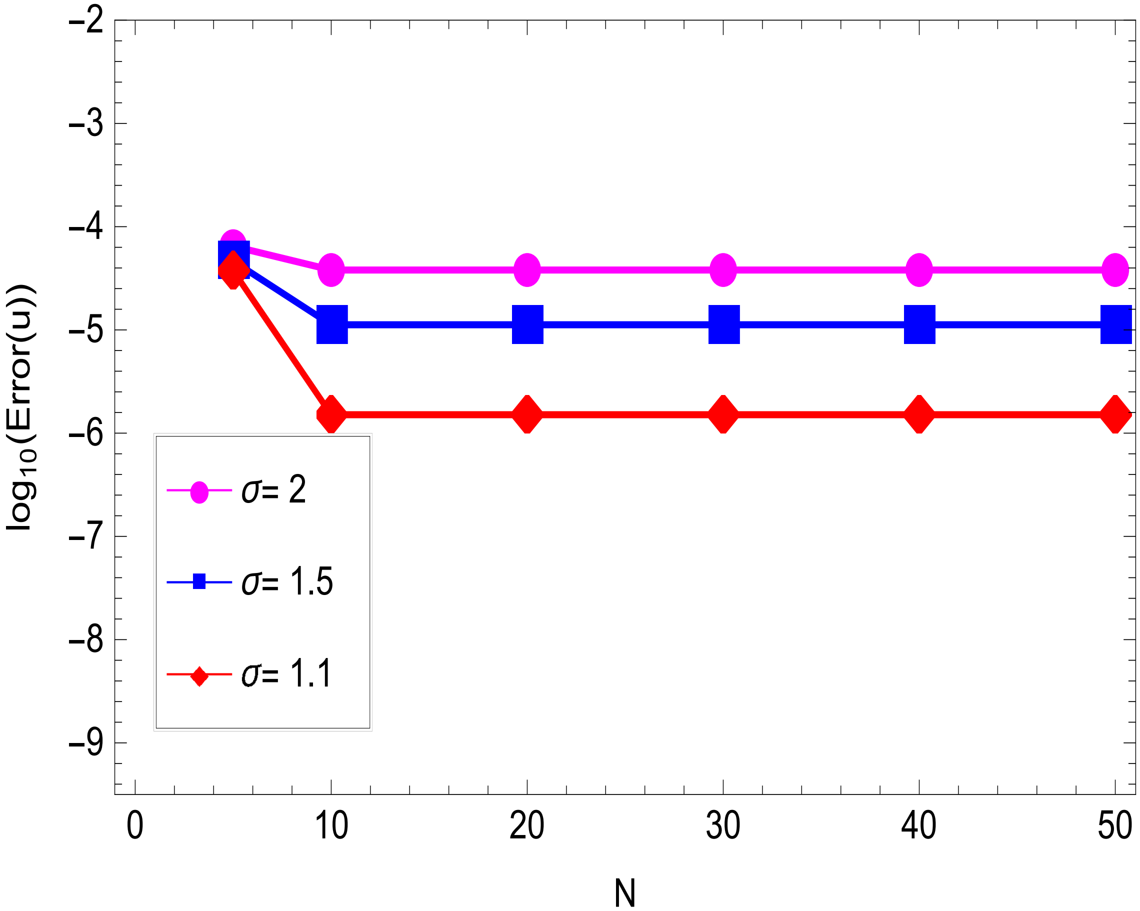

Figure 1: The rate of Convergence in the spatial direction with different σ at $\tau = 1/1600$.

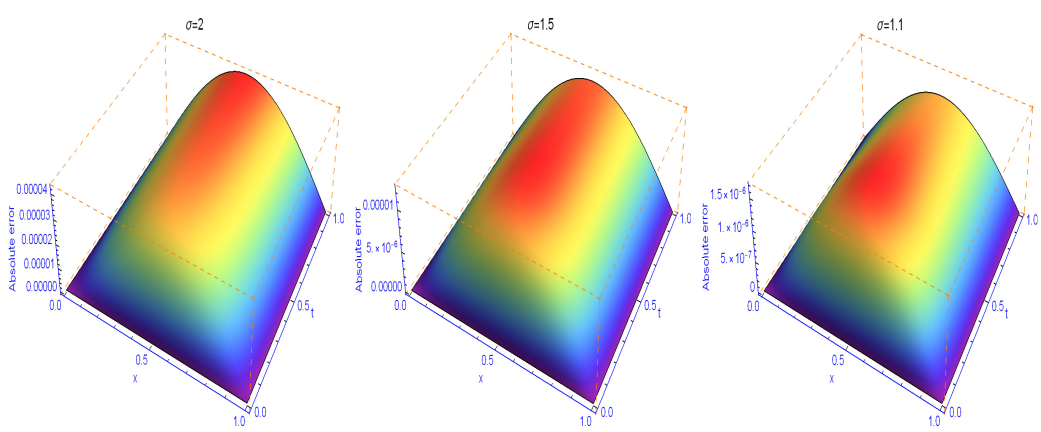

Figure 2: The absolute error function with $\sigma = 1.1,\ 1.5,\ 2$, $N = 50$ and $\tau = 1/2000$.

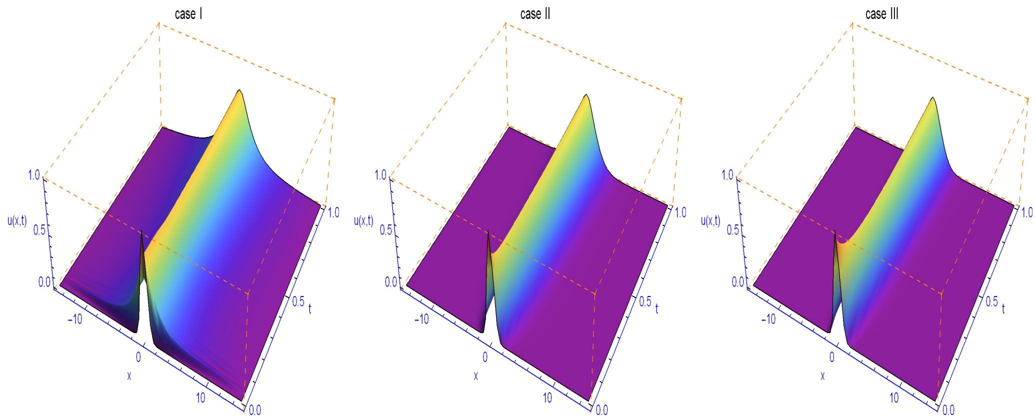

Figure 3: Behaviour of the solutions to model (4.31) with $\lambda = 0$, $N = 100$ and $\tau = 0.003$.

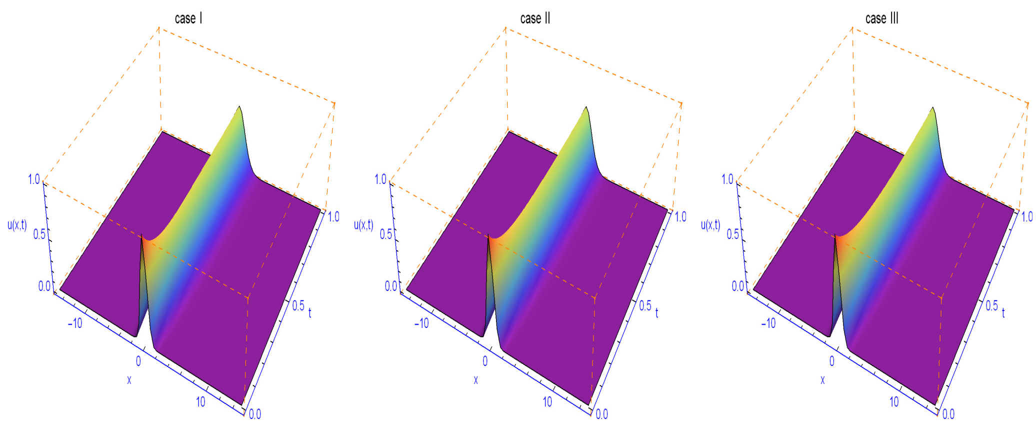

Figure 4: Behaviour of the solutions to model (4.31) with $\lambda = 1$, $N = 100$ and $\tau = 0.003$.

5. Conclusions and remarks

A numerical framework based on a combination of $L1$-difference approximation and Galerkin Legendre spectral method is introduced to solve variable order subdiffusion equation with delay. The resulted scheme is linear despite the nonlinearity of the problem under consideration. Standard techniques based on mathematical induction and discrete energy estimates are handled to prove the unconditional estimates of convergence and stability. For the sake of clearness, some numerical experiments are constructed to show the scheme's efficiency by examining the convergence order in both temporal and spatial directions, the absolute error for different time-fractional orders, and the numerical solution's physical behaviour.

Acknowledgments

The first and fourth authors were supported by the Nazarbayev University Program 091019CRP2120. K. Van Bockstal is supported by a postdoctoral fellowship of the Research Foundation - Flanders (106016/12P2919N). A. S. Hendy wishes to acknowledge the support of the RSF grant, project 22-21-00075.

Data availability statement

The data used to support the findings of this study are included within the article.

Ethics declarations: Conflict of interest

All authors declare that they have no conflict of interest.

References

[1] H Sun, A Chang, Y Zhang, and W Chen. A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fractional Calculus and Applied Analysis*, 22(1):27–59, 2019.

[2] A Burlon, G Alotta, M Di Paola, and G Failla. An original perspective on variable-order fractional operators for viscoelastic materials. *Meccanica*, 56(4):769–784, 2021.

[3] R M Ganji, H Jafari, and D Baleanu. A new approach for solving multi variable orders differential equations with Mittag–Leffler kernel. *Chaos, Solitons & Fractals*, 130:109405, 2020.

[4] N Kadkhoda, H Jafari, and R M Ganji. A numerical solution of variable order diffusion and wave equations. *International Journal of Nonlinear Analysis and Applications*, 12(1):27–36, 2021.

[5] R M Ganji and H Jafari. A numerical approach for multi-variable orders differential equations using Jacobi polynomials. *International Journal of Applied and Computational Mathematics*, 5(2):1–9, 2019.

[6] V R Hosseini, M Koushki, and W-N Zou. The meshless approach for solving 2D variable-order time-fractional advection–diffusion equation arising in anomalous transport. *Engineering with computers*, pages 1–19, 2021.

[7] V R Hosseini, F Yousefi, and W-N Zou. The numerical solution of high dimensional variable-order time fractional diffusion equation via the singular boundary method. *Journal of Advanced Research*, 32:73–84, 2021.

[8] A Babaei, H Jafari, and S Banihashemi. Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method. *Journal of Computational and Applied Mathematics*, 377:112908, 2020.

[9] R M Ganji, H Jafari, and S Nemati. A new approach for solving integro-differential equations of variable order. *Journal of Computational and Applied Mathematics*, 379:112946, 2020.

[10] L E S Ramirez and C F M Coimbra. A variable order constitutive relation for viscoelasticity. *Annalen der Physik*, 16(7-8):543–552, 2007.

[11] G Scarpi. Sulla possibilita di un modello reologico intermedio di tipo evolutivo. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, 52(6):912–917, 1972.

[12] R Garrappa, A Giusti, and F Mainardi. Variable-order fractional calculus: A change of perspective. *Communications in Nonlinear Science and Numerical Simulation*, 102:105904, 2021.

[13] S Patnaik, J P Hollkamp, and F Semperlotti. Applications of variable-order fractional operators: a review. *Proceedings of the Royal Society A*, 476(2234):20190498, 2020.

[14] S Samko. Fractional integration and differentiation of variable order: an overview. *Nonlinear dynamics*, 71(4):653–662, 2013.

[15] M D Ortigueira, D Valério, and J T Machado. Variable order fractional systems. *Communications in Nonlinear Science and Numerical Simulation*, 71:231–243, 2019.

[16] R Metzler and J Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics reports*, 339(1):1–77, 2000.

[17] Y Luchko. Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. *Fractional Calculus and Applied Analysis*, 15(1):141–160, 2012.

[18] N Kopteva. Error analysis of the L1 method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions. *Mathematics of Computation*, 88(319):2135–2155, 2019.

[19] D A Benson, R Schumer, M M Meerschaert, and S W Wheatcraft. Fractional dispersion, Lévy motion, and the MADE tracer tests. *Transport in porous media*, 42(1):211–240, 2001.

[20] M Abbaszadeh and M Dehghan. Meshless upwind local radial basis function-finite difference technique to simulate the time-fractional distributed-order advection–diffusion equation. *Engineering with computers*, 2021(37):873–889, 2012.

[21] M Abbaszadeh and M Dehghan. A finite-difference procedure to solve weakly singular integro partial differential equation with space-time fractional derivatives. *Engineering with computers*, 37(2021):2173–2182, 2021.

[22] A S Hendy, M A Zaky, and D Suragan. Discrete fractional stochastic Grönwall inequalities arising in the numerical analysis of multi-term fractional order stochastic differential equations. *Mathematics and Computers in Simulation*, 193:269–279, 2022.

[23] A S Hendy, M A Zaky, R M Hafez, and R H De Staelen. The impact of memory effect on space fractional strong quantum couplers with tunable decay behavior and its numerical simulation. *Scientific Reports*, 11(1):1–15, 2021.

[24] A H Bhrawy and M A Zaky. An improved collocation method for multi-dimensional space–time variable-order fractional Schrödinger equations. *Applied Numerical Mathematics*, 111:197–218, 2017.

[25] A H Bhrawy and M A Zaky. Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations. *Computers & Mathematics with Applications*, 73(6):1100–1117, 2017.

[26] B Baeumer, M Kovács, M M Meerschaert, and H Sankaranarayanan. Boundary conditions for fractional diffusion. *Journal of Computational and Applied Mathematics*, 336:408–424, 2018.

[27] J Jia, H Wang, and X Zheng. A preconditioned fast finite element approximation to variable-order time-fractional diffusion equations in multiple space dimensions. *Applied Numerical Mathematics*, 163:15–29, 2021.

[28] H Wang and X Zheng. Wellposedness and regularity of the variable-order time-fractional diffusion equations. *Journal of Mathematical Analysis and Applications*, 475(2):1778–1802, 2019.

[29] D Tavares, R Almeida, and D F M Torres. Caputo derivatives of fractional variable order: numerical approximations. *Communications in Nonlinear Science and Numerical Simulation*, 35:69–87, 2016.

[30] S Chen, F Liu, and K Burrage. Numerical simulation of a new two-dimensional variable-order fractional percolation equation in non-homogeneous porous media. *Computers & Mathematics with Applications*, 68(12):2133–2141, 2014.

[31] F Zeng, Z Zhang, and G E Karniadakis. A generalized spectral collocation method with tunable accuracy for variable-order fractional differential equations. *SIAM Journal on Scientific Computing*, 37(6):A2710–A2732, 2015.

[32] M Zayernouri and G E Karniadakis. Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. *Journal of Computational Physics*, 293:312–338, 2015.

[33] T Zhao, Z Mao, and G E Karniadakis. Multi-domain spectral collocation method for variable-order nonlinear fractional differential equations. *Computer Methods in Applied Mechanics and Engineering*, 348:377–395, 2019.

[34] X Zheng and H Wang. Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions. *IMA Journal of Numerical Analysis*, 41(2):1522–1545, 2021.

[35] X Zheng and H Wang. A hidden-memory variable-order time-fractional optimal control model: Analysis and approximation. *SIAM Journal on Control and Optimization*, 59(3):1851–1880, 2021.

[36] J Jia, H Wang, and X Zheng. A fast collocation approximation to a two-sided variable-order space-fractional diffusion equation and its analysis. *Journal of Computational and Applied Mathematics*, 388:113234, 2021.

[37] H-K Pang and H-W Sun. A fast algorithm for the variable-order spatial fractional advection-diffusion equation. *Journal of Scientific Computing*, 87(1):1–28, 2021.

[38] L Wei and Y Yang. Optimal order finite difference/local discontinuous Galerkin method for variable-order time-fractional diffusion equation. *Journal of Computational and Applied Mathematics*, 383:113129, 2021.

[39] L Wei, S Zhai, and X Zhang. Error estimate of a fully discrete local discontinuous Galerkin method for variable-order time-fractional diffusion equations. *Communications on Applied Mathematics and Computation*, pages 1–15, 2020.

[40] H Liu, A Cheng, and H Wang. A parareal finite volume method for variable-order time-fractional diffusion equations. *Journal of Scientific Computing*, 85(1):1–27, 2020.

[41] X-M Gu, H-W Sun, Y-L Zhao, and X Zheng. An implicit difference scheme for time-fractional diffusion equations with a time-invariant type variable order. *Applied Mathematics Letters*, 120:107270, 2021.

[42] K Van Bockstal. Existence of a unique weak solution to a non-autonomous time-fractional diffusion equation with space-dependent variable order. *Advances in Difference Equations*, 2021(1):1–43, 2021.

[43] V G Pimenov and A S Hendy. Numerical studies for fractional functional differential equations with delay based on BDF-type shifted Chebyshev approximations. In *Abstract and Applied Analysis*, volume 2015. Hindawi, 2015.

[44] M A Zaky, A S Hendy, and J E Macías-Díaz. Semi-implicit Galerkin–Legendre spectral schemes for nonlinear time-space fractional diffusion–reaction equations with smooth and nonsmooth solutions. *Journal of Scientific Computing*, 82(1):1–27, 2020.

[45] A S Hendy and M A Zaky. Combined Galerkin spectral/finite difference method over graded meshes for the generalized nonlinear fractional Schrödinger equation. *Nonlinear Dynamics*, 103(3):2493–2507, 2021.

[46] C F Lorenzo and T T Hartley. Variable order and distributed order fractional operators. *Nonlinear dynamics*, 29(1):57–98, 2002.

[47] P Zhuang, F Liu, V Anh, and Ian Turner. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM Journal on Numerical Analysis*, 47(3):1760–1781, 2009.

[48] X Zheng and H Wang. An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation. *SIAM Journal on Numerical Analysis*, 58(5):2492–2514, 2020.

[49] X Zheng and H Wang. A time-fractional diffusion equation with space-time dependent hidden-memory variable order: analysis and approximation. *BIT Numerical Mathematics*, pages 1–29, 2021.

[50] H Wang and X Zheng. Analysis and numerical solution of a nonlinear variable-order fractional differential equation. *Advances in Computational Mathematics*, 45(5):2647–2675, 2019.

[51] X Zheng, Z Zhang, and H Wang. Analysis of a nonlinear variable-order fractional stochastic differential equation. *Applied Mathematics Letters*, 107:106461, 2020.

[52] Q Li, G Wang, and M Wei. Monotone iterative technique for time-space fractional diffusion equations involving delay. *Nonlinear Analysis: Modelling and Control*, 26(2):241–258, 2021.

[53] Z Liu, S Zeng, and Y Bai. Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications. *Fractional Calculus and Applied Analysis*, 19(1):188–211, 2016.

[54] S Shen, F Liu, J Chen, I Turner, and V Anh. Numerical techniques for the variable order time fractional diffusion equation. *Applied Mathematics and Computation*, 218(22):10861–10870, 2012.

[55] A A Alikhanov. Boundary value problems for the diffusion equation of the variable order in differential and difference settings. *Applied Mathematics and Computation*, 219(8):3938–3946, 2012.

[56] F Zeng, F Liu, C Li, K n Burrage, I Turner, and V Anh. A Crank–Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM Journal on Numerical Analysis*, 52(6):2599–2622, 2014.

[57] J Shen. Efficient spectral-Galerkin method I. direct solvers of second-and fourth-order equations using Legendre polynomials. *SIAM Journal on Scientific Computing*, 15(6):1489–1505, 1994.

[58] J Shen, T Tang, and L-L Wang. *Spectral methods: algorithms, analysis and applications*, volume 41. Springer Science & Business Media, 2011.

[59] Y Lin and C Xu. Finite difference/spectral approximations for the time-fractional diffusion equation. *Journal of computational physics*, 225(2):1533–1552, 2007.

[60] M A Zaky, A S Hendy, A A Alikhanov, and V G Pimenov. Numerical analysis of multi-term time-fractional nonlinear subdiffusion equations with time delay: What could possibly go wrong? *Communications in Nonlinear Science and Numerical Simulation*, 96:105672, 2021.